Preparing for Midterm #1

MATH 316 — Linear Algebra II
Midterm: Friday, Feb 16, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W.

(c) Determine the orthogonal projection of that same vector onto $W^\perp$.

(d) Determine the QR decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(e) Determine a basis for the orthogonal complement $W^\perp$.

Solution.

(a) Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ be the vectors spanning W. We first construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ using Gram–Schmidt (and then normalize afterwards):

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \frac{3}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$
- $\boldsymbol{q}_3 = \boldsymbol{w}_3 - \frac{\boldsymbol{w}_3 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 - \frac{\boldsymbol{w}_3 \cdot \boldsymbol{q}_2}{\boldsymbol{q}_2 \cdot \boldsymbol{q}_2} \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$

Normalizing, we obtain the orthonormal basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ for W.

Comment. Alternatively, we could normalize the vectors during the Gram–Schmidt process. In general, this introduces square roots and therefore isn't advisable when working by hand.

(b) The orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W is

$$\frac{\boldsymbol{v} \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 + \frac{\boldsymbol{v} \cdot \boldsymbol{q}_2}{\boldsymbol{q}_2 \cdot \boldsymbol{q}_2} \boldsymbol{q}_2 + \frac{\boldsymbol{v} \cdot \boldsymbol{q}_3}{\boldsymbol{q}_3 \cdot \boldsymbol{q}_3} \boldsymbol{q}_3 = 6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{11}{18} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix}.$$

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(c) We can compute this as the error of the projection in part (b): $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}$.

Indeed, note that the resulting vector is in $W^\perp$ (because it is the error of the projection of $\boldsymbol{v}$ onto W), while the error is $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix}$, which we know is in W (after all, it's the projection of $\boldsymbol{v}$ onto W) and therefore orthogonal to $W^\perp$.

Comment. In general, if $\boldsymbol{v}_1$ is the orthogonal projection of $\boldsymbol{v}$ onto a space W, then $\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{v}_1$ is the projection of $\boldsymbol{v}$ onto $W^\perp$. (If this is not clear to you, notice that $\boldsymbol{v}_2$ is the error of projecting $\boldsymbol{v}$ onto W and therefore orthogonal to W; in other words, $\boldsymbol{v}_2$ is in $W^\perp$. On the other hand, the error of this second projection is $\boldsymbol{v} - \boldsymbol{v}_2 = \boldsymbol{v} - (\boldsymbol{v} - \boldsymbol{v}_1) = \boldsymbol{v}_1$ which is in W and, thus, orthogonal to $W^\perp$.)

Consequently, any vector $\boldsymbol{v}$ can be written in a unique way as $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with $\boldsymbol{v}_1$ in W and $\boldsymbol{v}_2$ in $W^\perp$. Here, $\boldsymbol{v}_1$ is the projection of $\boldsymbol{v}$ onto W and $\boldsymbol{v}_2$ is the projection of $\boldsymbol{v}$ onto $W^\perp$.

(d) From the first part, we know that $Q = \begin{bmatrix} 0 & 2/3 & -1/\sqrt{18} \\ 1 & 0 & 0 \\ 0 & 2/3 & -1/\sqrt{18} \\ 0 & 1/3 & 4/\sqrt{18} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 2/3 & 1/3 \\ -1/\sqrt{18} & 0 & -1/\sqrt{18} & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 3 & 5/3 \\ 0 & 0 & 2/\sqrt{18} \end{bmatrix}$.

(e) Clearly, $\dim W^\perp = 1$, so that $W^\perp$ is spanned by a single vector.

One way to determine vectors $W^\perp$ is to take any vector $\boldsymbol{v}$ (not in W) and project $\boldsymbol{v}$ onto W. The error of that projection then is in $W^\perp$.

We have already done that in part (c) of this problem: consequently, the vector $\begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}$ is a basis for $W^\perp$.

Alternatively. We can write $W = \operatorname{col}(A)$ where $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

It then follows from the fundamental theorem of linear algebra that $W^\perp = \operatorname{null}(A^T)$. That's a lot more work!

Problem 2.

(a) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ as $A = PDP^T$. (That is, find the matrices P and D.)

(b) Let A be a symmetric 2×2 matrix with 2-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\det(A) = -6$. Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) - 9 = (\lambda+8)(\lambda-2)$, and so A has eigenvalues $-8, 2$.

The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The -8-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Hence, if $P = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix}$, then $A = PDP^T$.

Important comment. Note that we were asked for a diagonalization of the form $A = PDP^T$ (which is possible, by the spectral theorem, because A is symmetric). For that, the matrix P must be orthogonal (that is, a square matrix with orthonormal columns). In particular, we must normalize its columns! (Otherwise, we only have the usual diagonalization $A = PDP^{-1}$.)

(b) Since $\det(A) = -6$ is the product of the eigenvalues, we find that the second eigenvalue is -3.

Since A is symmetric, the eigenspaces are orthogonal. Hence, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a -3-eigenvector.

Normalizing, a diagonalization of A is $A = PDP^T$ with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & \\ & -3 \end{bmatrix}$.

Important comment. Again, if we don't normalize and choose $P = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & \\ & -3 \end{bmatrix}$, then we only have a diagonalization of the form $A = PDP^{-1}$ (and not $A = PDP^T$).

Problem 3.

(a) Find the least squares solution to the system $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto the space $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}\right\}$?

(c) Determine the least squares line for the data points $(-2,1), (-1,0), (0,3), (2,1)$.

(d) Determine the projection matrix P for orthogonally projecting onto W.

Solution. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(a) We compute $A^TA = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$ and $A^T\boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, so the normal equations $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ are

$$\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

(b) The orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto W is $A\hat{\boldsymbol{x}} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix}$.

Check. The error $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 \\ -8 \\ 12 \\ -4 \end{bmatrix}$ is orthogonal to both $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

(c) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

Hence, the least squares line is $y = \frac{9}{7} + \frac{1}{7}x$.

(d) The projection matrix is $P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \frac{1}{35}\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 21 & 14 & 7 & -7 \\ 14 & 11 & 8 & 2 \\ 7 & 8 & 9 & 11 \\ -7 & 2 & 11 & 29 \end{bmatrix}$.

Problem 4. Trying to find a relation between the quantities x and y, we measure the values

x	1	2	3	4
y	2	5	9	17

(a) We expect that y can be predicted as a linear function of the form $a + bx$. Find the best estimate for the coefficients. ["best" in the least squares sense]

(b) What changes if we suppose that y can be predicted as a quadratic function of the form $a + bx + cx^2$? Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution.

(a) If we had $y = a + bx$ exactly, then we could find a, b by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_{\boldsymbol{y}}.$$

To find the least squares estimate, we solve the normal equations $A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \boldsymbol{y}$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \text{ and } A^T \boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}.$$

We solve $\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}$ to find $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 33 \\ 107 \end{bmatrix} = \begin{bmatrix} -4 \\ 49/10 \end{bmatrix}$.

Hence, $a = -4$ and $b = 4.9$.

(b) Again, if we had $y = a + bx + cx^2$ exactly, then we could find a, b, c by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_{\boldsymbol{y}}.$$

We find the best fit by instead computing a least squares solution.

Extra. Now, it becomes a bit painful by hand (ask Sage for help!). The normal equations $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \boldsymbol{y}$ are:

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \\ 375 \end{bmatrix}.$$

Solving this system, we find $a = 2.25$, $b = -1.35$ and $c = 1.25$.

Problem 5.

(a) Is it true that $A^T A$ is always symmetric?

(b) If the columns of A are orthogonal, what can you say about $A^T A$?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices A is it true that $A^{-1} = A^T$?

Solution.

(a) Yes, $A^T A$ is always symmetric: $(A^T A)^T = A^T (A^T)^T = A^T A$

(b) In that case, $A^T A$ is a diagonal matrix, and the diagonal entries are the squared norms of the columns of A.

For instance. If $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $A^T A = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$.

(c) If the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ were orthogonal, then this would be correct. However, they are not orthogonal. (Technically, we would only need the third vector to be orthogonal to the first two.)

Indeed, we can see that $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ cannot be the correct projection because the error would be $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, which is not orthogonal to $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(d) Orthogonal matrices.

[For a square matrix, $A^{-1} = A^T$ if and only if $A^T A = I$. Hence, $A^{-1} = A^T$ if and only if A is a square matrix with orthonormal columns. Such matrices are called orthogonal (a somewhat unfortunate terminology).]

Problem 6.

(a) We want to find values for the parameters a, b, c such that $y = a + bx + \frac{c}{x}$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

(b) We want to find values for the parameters a, b such that $y = (a + bx)e^x$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b]^T$ is a least squares solution.

(c) We want to find values for the parameters a, b, c such that $z = a + bx - c\sqrt{y}$ best fits some given points (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution.

(a) The equations $a + bx_i + c/x_i = y_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & 1/x_1 \\ 1 & x_2 & 1/x_2 \\ 1 & x_3 & 1/x_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{y}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

(b) The equations $(a + bx_i)e^{x_i} = y_i$ translate into the system:

$$\underbrace{\begin{bmatrix} e^{x_1} & x_1 e^{x_1} \\ e^{x_2} & x_2 e^{x_2} \\ e^{x_3} & x_3 e^{x_3} \\ \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{y}}$$

Of course, this is usually inconsistent. To find the best possible a, b we compute a least squares solution.

(c) The equations $a + bx_i - c\sqrt{y_i} = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & -\sqrt{y_1} \\ 1 & x_2 & -\sqrt{y_2} \\ 1 & x_3 & -\sqrt{y_3} \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

Problem 7. Let W be the subspace of $\mathbb{R}^4$ of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for W.

(b) Find a basis for the orthogonal complement $W^\perp$.

(c) Determine the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto $W^\perp$.

(d) Determine the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto W.

Solution. Note that $W = \text{null}(A)$ for the matrix $A = [\ 1\ \ 1\ \ 1\ \ -1\ ]$.

(a) A is already in RREF, so we can read off that $W = \text{null}(A)$ consists of the vectors $\begin{bmatrix} -s_1 - s_2 + s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$.

Hence, a basis for W is: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Recall that the orthogonal complement of $\text{null}(A)$ is $\text{row}(A)$.

Hence, a basis for $W^\perp$ is: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. (Note how this vector is indeed orthogonal to all basis vectors of W.)

(c) Since $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is an orthogonal basis for $W^\perp$, the projection is $\frac{\boldsymbol{b} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}} \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

(d) The easiest way is to observe that the projection of $\boldsymbol{b}$ onto W must be $\boldsymbol{b} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$.

Here, we used the projection from the previous part in the same way as we did in Problem 1(c).

Alternatively. Using the basis for W from the first part (which is not orthogonal), we can compute the orthogonal projection in different ways. For instance, we can solve the least squares system $\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. If $\hat{\boldsymbol{x}}$ is the least squares solution, then the orthogonal projection is $\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{\boldsymbol{x}}$. (Do this and compare!)

Yet another alternative way is to compute an orthogonal basis for W using Gram–Schmidt and to then project $\boldsymbol{b}$ by just computing dot products. (Do this as well and compare!)

Problem 8. Suppose that A is a 3×5 matrix of rank 3.

(a) For each of the four fundamental subspaces of A, state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\operatorname{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.

Solution.

(a) $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are subspaces of $\mathbb{R}^3$, while $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are subspaces of $\mathbb{R}^5$.

(b) $\dim \operatorname{col}(A) = 3$, $\dim \operatorname{row}(A) = 3$, $\dim \operatorname{null}(A) = 5 - 3 = 2$, $\dim \operatorname{null}(A^T) = 3 - 3 = 0$.

(c) $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are orthogonal complements of each other.

Also, $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are orthogonal complements of each other.

(d) Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2 - 2R_1 \Rightarrow R_2}{R_3 - 3R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & -3 & -8 & -8 \end{bmatrix} \overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\underset{\rightsquigarrow}{\overset{-\frac{1}{2}R_2 \Rightarrow R_2}{-2R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_1 - 3R_3 \Rightarrow R_1}{R_2 - \frac{5}{2}R_3 \Rightarrow R_2}} \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence, we can read off the bases:

$\operatorname{col}(A)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(Knowing that $\dim \operatorname{col}(A) = 3$, so that $\operatorname{col}(A) = \mathbb{R}^3$, we could have also just written down the standard basis.)

$\operatorname{row}(A)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \\ 1 \\ 4 \end{bmatrix}$.

$\operatorname{null}(A)$ consists of the vectors $\begin{bmatrix} -2s_1 - s_2 \\ s_1 \\ 0 \\ -s_2 \\ s_2 \end{bmatrix}$ and so has basis $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

$\operatorname{null}(A^T)$ has dimension 0 (contains only the zero vector), and so has an empty basis (consisting of 0 vectors).

(e) The rank is the number of pivots, which is indeed 3 (also equals $\dim \operatorname{col}(A)$ and $\dim \operatorname{row}(A)$).

We predicted all the dimensions accurately.