Fourier series

A **Fourier series** for a function $f(x)$ is a series of the form

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

You may have seen Fourier series in other classes before. Our goal here is to tie them in with what we have learned about orthogonality.

In these other classes, you would have seen formulas for the coefficients a_k and b_k. We will see where those come from.

Observe that the right-hand side combination of cosines and sines is 2π-periodic.

Let us consider (nice) functions on $[0, 2\pi]$.

Or, equivalently, functions that are 2π-periodic.

We know that a natural inner product for that space of functions is

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

Example 190. Show that $\cos(x)$ and $\sin(x)$ are orthogonal (in that sense).

Solution. $\langle\cos(x), \sin(x)\rangle = \int_0^{2\pi}\cos(t)\sin(t)\mathrm{d}t = \left[\frac{1}{2}(\sin(t))^2\right]_0^{2\pi} = 0$

In fact:

$\boxed{\text{All the functions } 1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots \text{ are orthogonal to each other!}}$

Moreover, they form a basis in the sense that every other (nice) function can be written as a (infinite) linear combination of these basis functions.

Example 191. What is the norm of $\cos(x)$?

Solution. $\langle\cos(x), \cos(x)\rangle = \int_0^{2\pi}\cos(t)\cos(t)\mathrm{d}t = \pi$

Why? There's many ways to evaluate this integral. For instance:

- integration by parts
- using a trig identity
- here's a simple way:
  - $\int_0^{2\pi}\cos^2(t)\mathrm{d}t = \int_0^{2\pi}\sin^2(t)\mathrm{d}t$ (cos and sin are just a shift apart)
  - $\cos^2(t) + \sin^2(t) = 1$
  - So: $\int_0^{2\pi}\cos^2(t)\mathrm{d}t = \frac{1}{2}\int_0^{2\pi}1\mathrm{d}x = \pi$

Hence, $\cos(x)$ is not normalized. It has norm $\|\cos(x)\| = \sqrt{\pi}$.

Similarly. The same calculation shows that $\cos(kx)$ and $\sin(kx)$ have norm $\sqrt{\pi}$ as well.

Example 192. How do we find, say, b_2?

Solution. Since the functions $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$, the term $b_2 \sin(2x)$ is the orthogonal projection of $f(x)$ onto $\sin(2x)$.

In particular, $b_2 = \frac{\langle f(x), \sin(2x)\rangle}{\langle \sin(2x), \sin(2x)\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}x$.

In conclusion:

A (nice) $f(x)$ on $[0, 2\pi]$ has the Fourier series

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

where

$$\begin{aligned} a_k = \frac{\langle f(x), \cos(kx)\rangle}{\langle \cos(kx), \cos(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t, \\ b_k = \frac{\langle f(x), \sin(kx)\rangle}{\langle \sin(kx), \sin(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)\mathrm{d}t, \\ a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t. \end{aligned}$$

The next example illustrates that we can likewise deal with intervals other than $[0, 2\pi]$ (or, equivalently, 2π-periodic functions).

The main observation is that, since $\cos(x)$ has period 2π, the scaled function $\cos\left(\frac{2\pi}{L}x\right)$ has period L.
As we are just scaling, it is not hard to see that the functions

$$1, \cos\left(\frac{2\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right), \cos\left(2\cdot\frac{2\pi}{L}x\right), \sin\left(2\cdot\frac{2\pi}{L}x\right), \cos\left(3\cdot\frac{2\pi}{L}x\right), \ldots$$

are still orthogonal to each other—now, adjusted for period L, with respect to the inner product

$$\langle f, g\rangle = \int_0^L f(t)g(t)\mathrm{d}t.$$

Example 193. Suppose that $f(x)$ is 5-periodic. Write down the first few terms of the Fourier series for $f(x)$ with undetermined coefficients. Spell out how to compute the coefficients of the sine functions.

Solution. The Fourier series for $f(x)$ is

$$f(x) = a_0 + a_1\cos\left(\frac{2\pi}{5}x\right) + b_1\sin\left(\frac{2\pi}{5}x\right) + a_2\cos\left(\frac{4\pi}{5}x\right) + b_2\sin\left(\frac{4\pi}{5}x\right) + a_3\cos\left(\frac{6\pi}{5}x\right) + \ldots$$

The coefficients b_n can be computed as

$$b_n = \frac{\left\langle f(x), \sin\left(\frac{2\pi}{5}nx\right)\right\rangle}{\left\langle \sin\left(\frac{2\pi}{5}nx\right), \sin\left(\frac{2\pi}{5}nx\right)\right\rangle} = \frac{\int_0^5 f(t)\sin\left(\frac{2\pi}{5}nt\right)\mathrm{d}t}{\int_0^5 \sin^2\left(\frac{2\pi}{5}nt\right)\mathrm{d}t} = \frac{2}{5}\int_0^5 f(t)\sin\left(\frac{2\pi}{5}nt\right)\mathrm{d}t.$$

For the final (optional) equality, we used that $\int_0^5 \sin^2\left(\frac{2\pi}{5}nt\right)\mathrm{d}t = \int_0^5 \cos^2\left(\frac{2\pi}{5}nt\right)\mathrm{d}t$ combined with $\cos^2 + \sin^2 = 1$ to conclude that the integral in the denominator must be $\frac{5}{2}$.