Example 185. Find the best approximation of $f(x)=\sqrt{x}$ (in the L^2 sense) on the interval $[0,1]$ using a function of the form $y=a+bx$.

Important observation. The orthogonal projection of $f:[0,1]\to\mathbb{R}$ onto $\text{span}\{1,x\}$ is not simply the projection onto 1 plus the projection onto x. That's because 1 and x are not orthogonal:

$$\langle 1,x\rangle=\int_0^1 t\,\mathrm{d}t=\frac{1}{2}\neq 0.$$

Solution. To find an orthogonal basis for $\text{span}\{1,x\}$, following Gram–Schmidt, we compute

$$x-\left(\begin{array}{c}\text{projection of}\\ x\text{ onto }1\end{array}\right)=x-\frac{\langle x,1\rangle}{\langle 1,1\rangle}1=x-\frac{1}{2}.$$

Hence, $1,x-\frac{1}{2}$ is an orthogonal basis for $\text{span}\{1,x\}$.
The orthogonal projection of $\sqrt{x}$ on $[0,1]$ onto $\text{span}\{1,x\}=\text{span}\left\{1,x-\frac{1}{2}\right\}$ therefore is

$$\frac{\langle\sqrt{x},1\rangle}{\langle 1,1\rangle}1+\frac{\left\langle\sqrt{x},x-\frac{1}{2}\right\rangle}{\left\langle x-\frac{1}{2},x-\frac{1}{2}\right\rangle}\left(x-\frac{1}{2}\right)=\frac{\int_0^1\sqrt{t}\,\mathrm{d}t}{\int_0^1 1\,\mathrm{d}t}+\frac{\int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t}{\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t}\left(x-\frac{1}{2}\right).$$

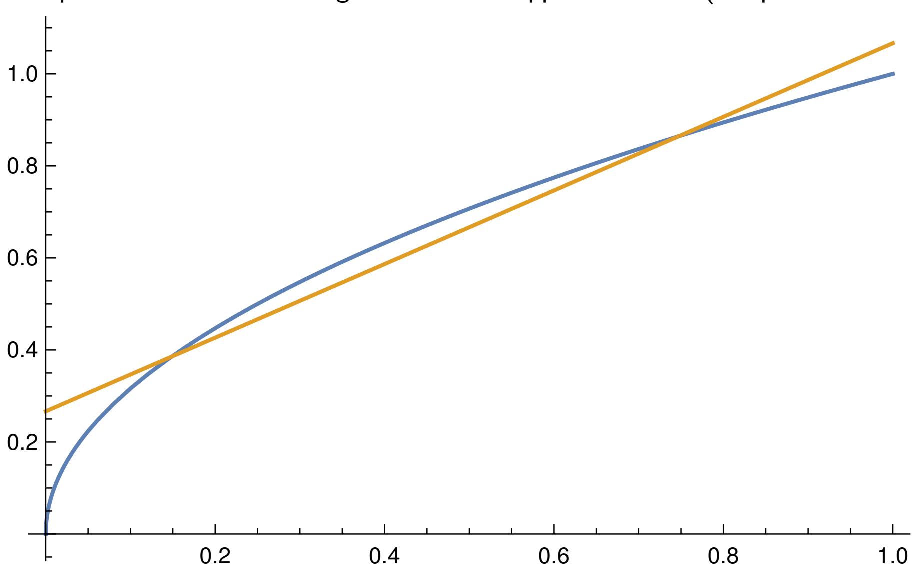
We compute the three new integrals:

$$\begin{aligned}\int_0^1\sqrt{t}\,\mathrm{d}t &= \left[\frac{2}{3}t^{3/2}\right]_0^1=\frac{2}{3}\\ \int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t &= \int_0^1\left(t^{3/2}-\frac{1}{2}t^{1/2}\right)\mathrm{d}t=\left[\frac{2}{5}t^{5/2}-\frac{1}{3}t^{3/2}\right]_0^1=\frac{2}{5}-\frac{1}{3}=\frac{1}{15}\\ \int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t &= \int_0^1\left(t^2-t+\frac{1}{4}\right)\mathrm{d}t=\left[\frac{1}{3}t^3-\frac{1}{2}t^2+\frac{1}{4}t\right]_0^1=\frac{1}{3}-\frac{1}{2}+\frac{1}{4}=\frac{1}{12}\end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_0^1\sqrt{t}\,\mathrm{d}t}{\int_0^1 1\,\mathrm{d}t}+\frac{\int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t}{\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t}\left(x-\frac{1}{2}\right)=\frac{2}{3}+\frac{12}{15}\left(x-\frac{1}{2}\right)=\frac{4}{5}x+\frac{4}{15}$$

The plot below confirms how good this linear approximation is (compare with the previous example):



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