

Example 167. Determine the SVD of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution. $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ has characteristic polynomial $(1 - \lambda)(2 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$.

The eigenvalues of $A^T A$ are $\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$.

$\frac{3 + \sqrt{5}}{2}$ -eigenvector $\begin{bmatrix} 2 \\ 1 + \sqrt{5} \end{bmatrix}$ and $\frac{3 - \sqrt{5}}{2}$ -eigenvector $\begin{bmatrix} 2 \\ 1 - \sqrt{5} \end{bmatrix}$.

It would be rather painful to continue with exact expressions, and that is not how applications typically proceed. Numerically:

- 2.618-eigenvector $\begin{bmatrix} 0.526 \\ 0.851 \end{bmatrix}$ and 0.382-eigenvector $\begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix}$. These eigenvectors are normalized, and it is now actually immediately obvious that they are orthogonal. (Of course, they had to be!)
- Hence, $\Sigma = \begin{bmatrix} \sqrt{2.618} & \\ & \sqrt{0.382} \end{bmatrix} = \begin{bmatrix} 1.618 & \\ & 0.618 \end{bmatrix}$ and $V = \begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix}$.
[We chose $\begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix}$ instead of $\begin{bmatrix} 0.851 \\ -0.526 \end{bmatrix}$, so that, for the resulting V , $\det V = +1$.]
- $\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{1.618} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.526 \\ 0.851 \end{bmatrix} = \begin{bmatrix} 0.851 \\ 0.526 \end{bmatrix}$
 $\mathbf{u}_2 = \frac{1}{\sigma_2} A \mathbf{v}_2 = \frac{1}{0.618} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix} = \begin{bmatrix} -0.526 \\ 0.851 \end{bmatrix}$.
Hence, $U = \begin{bmatrix} 0.851 & -0.526 \\ 0.526 & 0.851 \end{bmatrix}$. (Again, notice the obvious orthogonality!)

Comment. The matrix A itself has eigenvalues 1, 1, but the 1-eigenspace is only 1-dimensional. We are missing an eigenvector, which renders A not diagonalizable.

Comment. If we had continued symbolically, there are some magical simplifications like $\sqrt{\frac{3 + \sqrt{5}}{2}} = \frac{1 + \sqrt{5}}{2}$ going on. By the way, this is the golden ratio!

Sage. In Sage, the SVD is currently only implemented for floating point numbers (RDF is the real numbers as floating point numbers with double precision). Here's our computation:

```
Sage] A = matrix(RDF, [[1,1],[0,1]])
```

```
Sage] U,S,V = A.SVD()
```

```
Sage] U
```

$$\begin{bmatrix} 0.850650808352 & -0.525731112119 \\ 0.525731112119 & 0.850650808352 \end{bmatrix}$$

```
Sage] S
```

$$\begin{bmatrix} 1.61803398875 & 0.0 \\ 0.0 & 0.61803398875 \end{bmatrix}$$

```
Sage] V
```

$$\begin{bmatrix} 0.525731112119 & -0.850650808352 \\ 0.850650808352 & 0.525731112119 \end{bmatrix}$$

Example 168. (continued) The matrices U and V are rotation matrices. By what angle?

Why rotations? Recall that orthogonal matrices have determinant $+1$ or -1 .

Since $\det U = +1$ and $\det V = +1$, the orthogonal matrices U, V are rotations.

Solution. Being rotation matrices, each of them equals $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some angle θ .

To find the angle θ_V for V , we compute $\arccos(0.526) = 1.017$. This means that $\theta_V = 1.017$ or $\theta_V = 2\pi - 1.017$ (make a sketch of $\cos(\theta)$ if that's unclear!). Since $\sin(1.017) = 0.851$ (whereas $\sin(2\pi - 1.017) = -0.851$), we conclude that V is a rotation by $\theta_V = 1.017 = 58.3^\circ$. Keep that angle in mind for the next example!

Likewise, U is a rotation by $\theta_U = 0.554 = 31.7^\circ$.

Comment. The two angles add up to 90° . That's a consequence of the (atypical) fact that the matrices U and V have essentially the same entries.

Example 169. Explain the geometric meaning of the SVD in the previous example.

- The map $\mathbf{x} \mapsto A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ sends the (orthogonal) grid spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the (nonorthogonal) grid spanned by $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Make a sketch! The two grids are overlaid in the first plot on the next page.

- Likewise, for instance, the (orthogonal) grid spanned by $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the 45° degree rotated version of the previous grid) is sent to the (again, nonorthogonal) grid spanned by $\frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Make a sketch! The two grids are overlaid in the second plot on the next page.

- Can we find an orthogonal grid which is sent to another orthogonal grid by A ?

Solution. Yes! The SVD $A = U\Sigma V^T$ is equivalent to $AV = U\Sigma$. That is, $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$.

In other words, the orthogonal grid spanned by $\mathbf{v}_1, \mathbf{v}_2$ is sent to the orthogonal grid spanned by $\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2$. As we observed earlier, the grid spanned by $\mathbf{v}_1, \mathbf{v}_2$ is the 58.3° degree rotated version of the standard grid)

While the input grid consists of little squares, the output grid consists of rectangles with sides σ_1, σ_2 .

Make a sketch! The two grids are overlaid in the third plot on the next page.

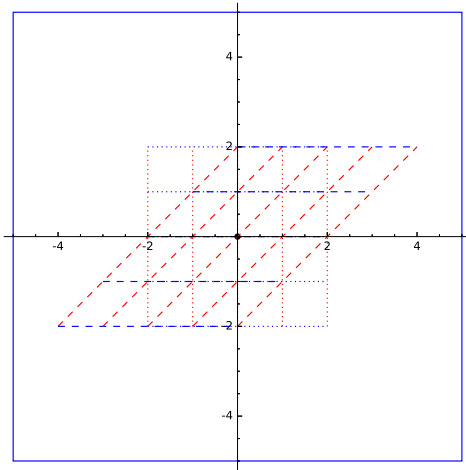
The following Sage code prepares the plots on the next page. Even if you have no coding background, can you see, roughly, what is happening?

```
Sage] def grid_lines(v1, v2, n, args={}):
    lines = Graphics()
    for i in [-n..n]:
        lines += line([i*v1-n*v2, i*v1+n*v2], color='red', **args)
        lines += line([i*v2-n*v1, i*v2+n*v1], color='blue', **args)
    return lines
```

```
Sage] def svd_rotate(angle):
    A = matrix([[1,1],[0,1]])
    t = angle*2*pi/360
    R = matrix([[cos(t),-sin(t)],[sin(t),cos(t)]])
    G1 = grid_lines(R*vector([1,0]), R*vector([0,1]), 2, {'linestyle':'-'})
    G2 = grid_lines(A*R*vector([1,0]), A*R*vector([0,1]), 2, {'linestyle':'--'})
    B = polygon([(-5,-5), (-5,5), (5,5), (5,-5)], fill=False)
    O = point((0,0), pointsize=30,color='black')
    return B+O+G1+G2
```

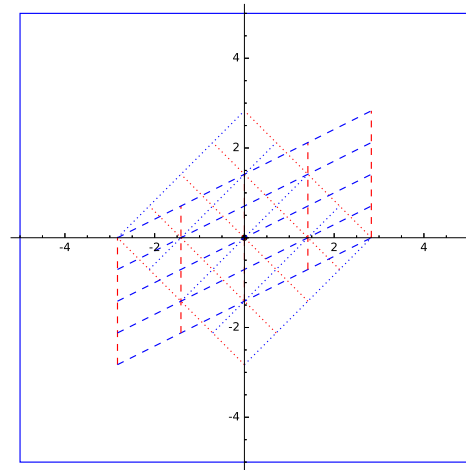
Grid spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (dotted), and grid spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (dashed):

Sage] `svd_rotate(angle = 0)`



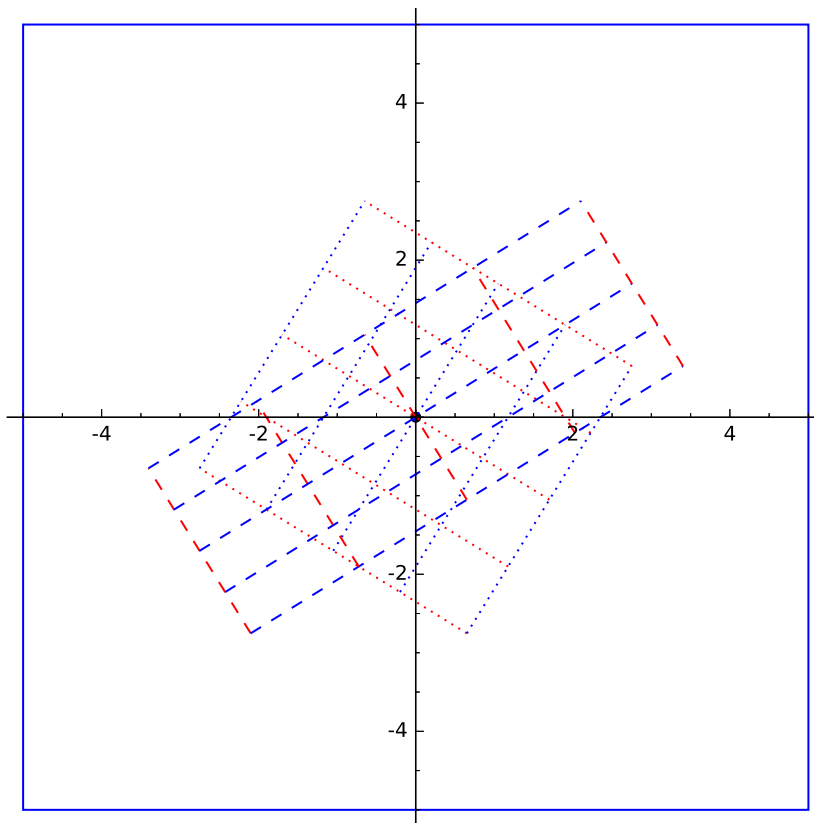
Grid spanned by $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (dotted), and grid spanned by $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (dashed):

Sage] `svd_rotate(angle = 45)`



Finally, here is the special situation (given by the SVD!) which shows an orthogonal grid (rotated by 58.3° degree) that is sent to another orthogonal grid (rotated by 31.7° degree):

Sage] `svd_rotate(angle = 58.3)`



For more pictures and detailed comments see the beautiful article:
<http://www.ams.org/samplings/feature-column/fcarc-svd>