Example 152. Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- We first diagonalize $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
  - $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so the eigenvalues are ± 1.
  - The 1-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}}_{=\begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}}\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix} \end{aligned}$$

Check. Indeed, $y_1 = \frac{1}{2}(e^t + e^{-t})$ and $y_2 = \frac{1}{2}(e^t - e^{-t})$ satisfy the system of differential equations $y_1' = y_2$ and $y_2' = y_1$ as well as the initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

Comment. You have actually met these functions in Calculus! $y_1 = \cosh(t)$ and $y_2 = \sinh(t)$. Check out the next example for the connection to $\cos(t)$ and $\sin(t)$.

Example 153.

(a) Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Show that $\boldsymbol{y} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ solves the same IVP. What do you conclude?

Solution.

(a) $A = PDP^{-1}$ with $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

The system is therefore solved by:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} \\ -e^{-it} \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} ie^{it} + ie^{-it} \\ e^{it} - e^{-it} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix} \end{aligned}$$

(b) Clearly, $\boldsymbol{y}(0) = \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. On the other hand, $y_1' = -\sin(t) = -y_2$ and $y_2' = \cos(t) = y_1$, so that $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$. Since the solution to the IVP is unique, it follows that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix}$.

We have just discovered **Euler's identity**!

Theorem 154. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Another short proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Rotation matrices

Example 155. Write down a 2×2 matrix Q for rotation by angle θ in the plane.

Comment. Why should we even be able to represent something like rotation by a matrix? Meaning that $Q\boldsymbol{x}$ should be the vector $\boldsymbol{x}$ rotated by θ. Recall from Linear Algebra I that every **linear map** can be represented by a matrix. Then think about why rotation is a linear map.

Solution. We can determine Q by figuring out $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the first column of Q) and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the second column of Q).

Since $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, we conclude that $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Comment. Note that we don't need previous knowledge of $\cos$ and $\sin$. We could have introduced these trig functions on the spot.

Comment. Note that it is geometrically obvious that Q is orthogonal. (Why?)

It is clear that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = 1$. Noting that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = \cos^2\theta + \sin^2\theta$, we have rediscovered Pythagoras.

Advanced comment. Actually, every orthogonal 2×2 matrix Q with $\det(Q) = 1$ is a rotation by some angle θ. Orthogonal matrices with $\det(Q) = -1$ are reflections.

Example 156. As in the previous example, let Q_θ be the 2×2 matrix for rotation by angle θ in the plane. What is $Q_\alpha Q_\beta$?

Solution. Note that $Q_\alpha Q_\beta \boldsymbol{x}$ first rotates $\boldsymbol{x}$ by angle β and then by angle α. For geometric reasons, it is obvious that this is the same as if we rotated $\boldsymbol{x}$ by $\alpha + \beta$. It follows that $Q_\alpha Q_\beta = Q_{\alpha+\beta}$.

Comment. This allows us to derive interesting trig identities:

$$\begin{aligned} Q_\alpha Q_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \ldots \\ \ldots & \ldots \end{bmatrix} \\ Q_{\alpha+\beta} &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \end{aligned}$$

It follows that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Comment. If we set $\beta = \alpha$, this simplifies to $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$, the double angle formula that you have probably used countless times in Calculus.

Comment. Similarly, we find an identity for $\sin(\alpha+\beta)$. Spell it out!

More on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at Example 89.

- The **absolute value** of the complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.
  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 157.

- For every matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.)

Example 158. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1+i & 2-3i \end{bmatrix} \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} = |1-i|^2 + |2+3i|^2 = 2 + 13$. Hence, $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 159. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Example 160. What is $\frac{1}{2+3i}$?

Solution. $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$.

In general. $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|z|^2}$