Review.

- Let A be $n \times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Then, $\frac{\mathrm{d}}{\mathrm{d}t} e^{At} = Ae^{At}$.

Why? $\frac{\mathrm{d}}{\mathrm{d}t} e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.
- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

Example 147. The matrix exponential shares many other properties of the usual exponential:

- $e^Ae^B = e^{A+B} = e^Be^A$ if $AB = BA$

Why the condition $\boldsymbol{AB} = \boldsymbol{BA}$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

$$e^Ae^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

Why? That actually follows from the previous property.

Example 148. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{At} = I + At + \frac{t^2}{2!}A^2 + \ldots = I + At = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

Example 149. Compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$.

Solution. Note that $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$.

Hence, $e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots = I + At + \frac{1}{2}A^2t^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ & 0 & t \\ & & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & t^2 \\ & 0 & 0 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix}$.

Example 150. Compute e^{At} for $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and N commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!}N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ & e^{2t} \end{bmatrix}$.

Advanced. Can you show that $A^n = \begin{bmatrix} 2^n & n2^{n-1} \\ & 2^n \end{bmatrix}$?

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Example 151. Solve the differential equation

$$\boldsymbol{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_{A} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_0}$$

Solution. Repeating the work in the previous example, the solution to the differential equation is

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 \\
&= e^{2It+Nt}\boldsymbol{y}_0 \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e^{2It}e^{Nt}\boldsymbol{y}_0 \qquad \text{(because } 2It \text{ and } Nt \text{ commute)} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots\right)\boldsymbol{y}_0 \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}(1 + Nt)\boldsymbol{y}_0 \qquad \text{(because } N^2 = \boldsymbol{0}) \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}.
\end{aligned}$$

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t\,\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t}, whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for every linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t\cos(2t)$ and $t\sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

https://en.wikipedia.org/wiki/Resonance

Understanding when resonance occurs is of crucial importance for practical applications.