Example 138. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0)=2,\ y_1'(0)=3,\ y_2(0)=-1,\ y_2'(0)=1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

The Jordan normal form

Note that we currently only know how to compute e^{At} when A is diagonalizable. Our next goal is to be able to compute the matrix exponential for all matrices.

Example 139. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$.

Solution. The eigenvalues of A are $4, 4$.

However, the 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 140. A **λ-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).
As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 141. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.

Armin Straub
straub@southalabama.edu

Example 142. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues 4, $4,4$?

Solution. $\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$

The dimension of the 4-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you know from diagonalization).

Example 143.

(a) What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $3,3,3$?

(b) What are the possible Jordan normal forms of a 4×4 matrix with eigenvalues $3,3,3,3$?

(c) What if the matrix is 5×5 and has eigenvalues $4,4,3,3,3$?

Solution.

(a) $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 3 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you known from diagonalization).

(b) Now, there are 5 possibilities:

$\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 4, 3, 2, 2, 1, respectively.

(c) $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

Note that this is just all possible (namely, 3) Jordan normal forms of a 3×3 matrix with eigenvalues 3, $3,3$ combined with all possible (namely, 2) Jordan normal forms of a 2×2 matrix with eigenvalues $4,4$. In total, that makes $3\cdot 2=6$ possibilities.

Comment. Let $p(n)$ be the number of inequivalent Jordan normal forms of an $n\times n$ matrix with a single eigenvalue, n times repeated. We have seen that $p(2)=2$, $p(3)=3$, $p(4)=5$. Note that $p(n)$ is equal to the number of ways of writing n as an ordered sum of positive integers: for instance, $p(4)=5$ because $4=3+1=2+2=2+1+1=1+1+1+1$.

$p(n)$ is referred to as the **partition function** and, surprisingly, is a remarkably interesting mathematical object.

https://en.wikipedia.org/wiki/Partition_function_(number_theory)

Armin Straub
straub@southalabama.edu

Example 144. (summary of small cases)

(a) There are 2 possible Jordan normal forms of a 2×2 matrix with eigenvalues λ, λ.

Namely. $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$

(b) There are 3 possible Jordan normal forms of a 3×3 matrix with eigenvalues $\lambda, \lambda, \lambda$.

Namely. $\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

(c) There are 5 possible Jordan normal forms of a 4×4 matrix with eigenvalues $\lambda, \lambda, \lambda, \lambda$.

Namely. $\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$

Example 145. What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $3, 3, 7, 7, 7, 7$?

Solution. There are $2\cdot 5 = 10$ possible Jordan normal forms for such a matrix:

$$\begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & 1 \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & 1 \\ & & & & & 7 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & & \\ & & & & 7 & 1 \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & \\ & & & & & 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & & \\ & 3 & & & & \\ & & 7 & 1 & & \\ & & & 7 & 1 & \\ & & & & 7 & 1 \\ & & & & & 7 \end{bmatrix}$$

Example 146. How many different Jordan normal forms are there in the following cases?

(a) A 8×8 matrix with eigenvalues $1, 1, 2, 2, 2, 4, 4, 4$?

(b) A 11×11 matrix with eigenvalues $1, 1, 1, 2, 2, 2, 2, 4, 4, 4, 4$?

Solution.

(a) $2\cdot 3\cdot 3 = 18$ possible Jordan normal forms

(b) $3\cdot 5\cdot 5 = 75$ possible Jordan normal forms