Definition 127. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Why? As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{At} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right] \\ &= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}.\end{aligned}$$

Therefore, $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$ indeed solves the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization!

Theorem 128. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^DP^{-1}$.

Why? Recall that, if $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.

$$\begin{aligned}e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)P^{-1} = Pe^DP^{-1}\end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for every "nice" function f. Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n \geqslant 0} a_n x^n$.

More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 129. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 130. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $At = \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix}$, $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 131. (homework) Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Solution. (final solution only) $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$.

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Example 132. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, we diagonalize:

  For $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$. (That's homework!)

- We can then compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix} \end{aligned}$$

Comment. It is not necessary to compute $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$ (of course, you could do it, but that's more work).
Instead, recall that $A^{-1}\boldsymbol{b}$ is the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$. Here, solving $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we find $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Check. $\boldsymbol{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$