Example 116. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = C_1(1+\sqrt{6})^n + C_2(1-\sqrt{6})^n$ and we only need to figure out the values of C_1 and C_2.
Using the two initial conditions, we get two equations:
($a_0 =$) $C_1 + C_2 = 0$, ($a_1 =$) $C_1(1+\sqrt{6}) + C_2(1-\sqrt{6}) = 1$.
Solving, we find $C_1 = \frac{1}{2\sqrt{6}}$ and $C_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did previously in the case of the Fibonacci numbers: starting with the recursion matrix M, we compute its diagonalization $M = PDP^{-1}$. Multiplying out $PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't need the Binet-like formula for this conclusion. Just the eigenvalues and the observation that C_1 cannot be 0 are enough. [We cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{6})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Another brief look at projections (and reflections)

(projections) Suppose that M is the projection matrix for projecting onto a subspace W.

- The 1-eigenspace of M is W.
- The 0-eigenspace of M is $W^\perp$.

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get projected to themselves. But those are precisely the vectors in W (recall that projecting a vector $\boldsymbol{v}$ onto W means producing the vector in W that is closest to $\boldsymbol{v}$). Can you likewise spell out the situation for the 0-eigenspace?

Note that M cannot have further eigenvalues (because the dimensions of W and $W^\perp$ already add up to the dimension of the space that we are working in).

Because the eigenvalues of M are real and the eigenspaces are orthogonal, the matrix M has a diagonalization of the form $M = PDP^T$ (make sure you can explain why!) which implies that M is symmetric (that's because $M^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = M$).

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Example 117. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^{-1}$.

(b) Diagonalize A as $A = PDP^T$.

Comment. This gives us yet another way to get our hands on projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

[Can you see that this is not really a "new" way of computing projection matrices? In particular, note that, if Q is the matrix P with the third column omitted, then $A = QQ^T$.]

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).

We already have a basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 0 & -1/4 \\ 0 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (of the 1-eigenspace), we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{17} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{17} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$.

Next, we normalize the vectors $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{17} \begin{bmatrix} -4 \\ 34 \\ 16 \end{bmatrix}$, $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{357}} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$, $\frac{1}{\sqrt{21}} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4/\sqrt{17} & -2/\sqrt{357} & -1/\sqrt{21} \\ 0 & 17/\sqrt{357} & -2/\sqrt{21} \\ 1/\sqrt{17} & 8/\sqrt{357} & 4/\sqrt{21} \end{bmatrix}$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{21} \begin{bmatrix} 20 & -2 & 4 \\ -2 & 17 & 8 \\ 4 & 8 & 5 \end{bmatrix}$ as in Example 55.

Example 118. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional). Note that we are lucky and already have an orthogonal basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal. Like any projection matrix, A is symmetric.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.