Application: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing? Have a look at ratios of Fibonacci numbers:
$\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, $\frac{34}{21} \approx 1.619$, ...

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 108 below.

The crucial insight is the following simple observation:

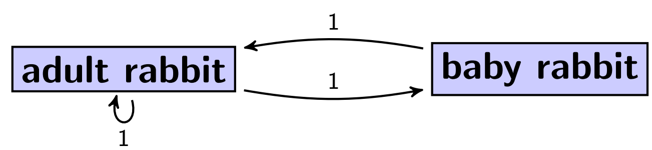
$$F_{n+2} = F_{n+1} + F_n \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Example 107. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these features might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let a_t be the number of adult rabbit pairs after t months. Likewise, b_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} a_t + b_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

Everything we observe here about Fibonacci numbers holds for other sequences that satisfy similar recursion equations.

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Theorem 108. **(Binet's formula)** $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof.

- We already observed that thee recurrence $F_{n+2} = F_{n+1} + F_n$ translates into $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $M = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = M^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} &= PD^nP^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix}\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix} \end{aligned}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}}\lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \xrightarrow{n\to\infty} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 109. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution. $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$

Important. If we write $\boldsymbol{a}_n = \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$, then this is simply $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

In particular, it follows that $\boldsymbol{a}_n = M^n\boldsymbol{a}_0$.

If we compute M^n, then this produces an explicit formula for a_n (the third entry of $\boldsymbol{a}_n$). This formula is called a **Binet-like formula** (in the case of the Fibonacci numbers, this is precisely the classical Binet formula).

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