Powers of matrices

Example 95. (warmup) Consider $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

- What are the eigenspaces?
- What are A^{-1} and A^{100}? What is A^n?

Solution.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a -2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector. In other words, the -2-eigenspace is $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and the 3-eigenspace is $\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.
- $A^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} (-2)^{100} & 0 \\ 0 & 3^{100} \end{bmatrix} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$. In general, $A^n = \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix}$.

Comment. Algebraically, the map $\boldsymbol{v} \mapsto A\boldsymbol{v}$ looks very simple. However, notice that it is not so easy to say what happens to, say, $\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of the vector $\boldsymbol{v}$ is scaled by -2, the other part is scaled by 3.

Example 96. If A has λ-eigenvector $\boldsymbol{v}$, then what can we say about A^2?

Solution. A^2 has λ^2-eigenvector $\boldsymbol{v}$.
[Indeed, $A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$. This is even easier in words: multiplying $\boldsymbol{v}$ with A has the effect of scaling it by λ; hence, multiplying it with A^2 scales it by λ^2.]

Important comment. Similarly, A^{100} has λ^{100}-eigenvector $\boldsymbol{v}$.

Example 97. If a matrix A can be diagonalized as $A = PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. Likewise, $A^n = PD^nP^{-1}$.
[The point being that D^n is trivial to compute because D is diagonal.]

In particular. $A^{-1} = PD^{-1}P^{-1}$

Important comment. In the previous example, we observed that, if A has λ-eigenvector $\boldsymbol{v}$, then A^n has λ^n-eigenvector $\boldsymbol{v}$. Note that this is also expressed in $A^n = PD^nP^{-1}$, because the latter is a diagonalization of A^n. The diagonalization shows that A^n and A have the same eigenvectors (since we can use the same matrix P) and that the eigenvalues of A^n are the n-th powers of the eigenvalues of A (which are the entries of the diagonal matrix D).

(computing matrix powers) If A is a square matrix with diagonalization $A = PDP^{-1}$, then

$$A^n = PD^nP^{-1}.$$

Example 98. Let $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$. Compute A^n.

Solution. First, we diagonalize: $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 10 & \\ & 5 \end{bmatrix}$. (Fill in the details!)

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & \\ & 5^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & 10^n \\ -4 \cdot 5^n & 1 \cdot 5^n \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$$

Check. Verify the cases $n = 0$ ($A^0 = I$) and $n = 1$.

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Example 99. (extra) Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$. Determine A^n.

Solution. We first repeat our work from Example 17 to find a diagonalization of A:
By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda)-2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

- $\lambda = 5$: $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$
- $\lambda = 2$: $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

We therefore have the diagonalization $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

[Keep in mind that other choices for P and D exist.]

With some labor (do it!), we find $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

It follows that

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2\cdot 5^n & 0 & -2^n \\ 2\cdot 5^n & 2^n & 0 \\ 5^n & 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -2 \\ -1 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2\cdot 5^n + 2^n & 0 & 2\cdot 5^n - 2\cdot 2^n \\ 2\cdot 5^n - 2\cdot 2^n & 3\cdot 2^n & 2\cdot 5^n - 2\cdot 2^n \\ 5^n - 2^n & 0 & 5^n + 2\cdot 2^n \end{bmatrix}. \end{aligned}$$

Check. Notice that it is particularly easy to verify the cases $n = 0$ ($A^0 = I$) and $n = 1$.