Review. A matrix A has orthonormal columns $\iff$ $A^T A = I$.

Example 78. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(Q)$?

Solution. Recall that, to project onto $\mathrm{col}(A)$, the projection matrix is $P = A(A^T A)^{-1} A^T$.

Since $Q^T Q = I$, to project onto $\mathrm{col}(Q)$, the projection matrix is $P = Q Q^T$.

Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q} \cdot \boldsymbol{b})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T \boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

Comment. In particular, if Q is not square, then $Q^T Q = I$ but $Q Q^T \neq I$. In some sense, $Q Q^T$ still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\mathrm{col}(Q)$ it does act like the identity for vectors in $\mathrm{col}(Q)$. (Vectors not in $\mathrm{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\mathrm{col}(Q)$.)

Example 79. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(A)$?

Solution. If A is an invertible $n \times n$ matrix, then $\mathrm{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\mathrm{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Definition 80. An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to be orthonormal, not just orthogonal.]

An $n \times n$ matrix Q is orthogonal $\iff$ $Q^T Q = I$

In other words, $Q^{-1} = Q^T$.

Review. Recall the following properties of determinants:

- $\det(AB) = \det(A)\det(B)$

  Comment. In fancy language, this means that the determinant is a group homomorphism between the group of (invertible) $n \times n$ matrices and (nonzero) complex numbers. Note that, on the left hand, we are multiplying matrices while, on the right hand, we are multiplying numbers. The key point is that it doesn't matter which multiplication we do: the two multiplications are compatible.

- $\det(A^{-1}) = \frac{1}{\det(A)}$

  Comment. Can you derive this from the previous property?

- $\det(A^T) = \det(A)$

  Comment. We are familiar with this in the context of cofactor expansion: it doesn't matter whether we expand by a column or by a row.

Example 81. What can we say about $\det(Q)$ if Q is orthogonal?

Solution. Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

Armin Straub
straub@southalabama.edu

Preview: The spectral theorem

Example 82. (review) In Example 17, we diagonalized $A=\begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ as $A=PDP^{-1}$.

We found that one choice for P and D is $P=\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Spell out what that tells us about A!

Solution. The diagonal entries $5,2,2$ of D are the eigenvalues of A.
The columns of P are corresponding eigenvectors of A.

- $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a 5-eigenvector of A (that is, $A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}=5\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$).
- The 2-eigenspace of A is 2-dimensional. A basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example 83. Diagonalize the **symmetric matrix** $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ as $A=PDP^{-1}$.

Review. Recall that a matrix A is symmetric if $A^T=A$.

Solution. We let Sage do the work for us:

```
Sage] A = matrix([[8,-6,2],[-6,7,-4],[2,-4,3]])
Sage] A.eigenmatrix_right()
```

$$\left(\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2} & 2 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}\right)$$

This ouput shows that A is diagonalizable as $A=PDP^{-1}$ with $D=\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P=\begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2} & 2 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$.

Just to make sure. This means that the eigenvalues of A are $15,3,0$ (the diagonal entries of D).

Moreover, we have that $\begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$ is a 15-eigenvector, $\begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ is a 3-eigenvector, and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a 0-eigenvector.

> **Important observation.** Note that the eigenspaces of A are orthogonal to each other here.
> The **spectral theorem** says that this is true for all symmetric matrices A.

Example 84. Diagonalize the symmetric matrix $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ as $A=PDP^T$.

Solution. By the previous example, we can diagonalize A as $\tilde{P}D\tilde{P}^{-1}$ with $\tilde{P}=\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $D=\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(To avoid fractions, we just scaled the first two columns of $\tilde{P}$, which are eigenvectors.)
Note that the columns of $\tilde{P}$ are orthogonal (this is due the spectral theorem). If we normalize them (they all have length $\sqrt{2^2+2^2+1}=3$), then we obtain the orthogonal matrix $P=\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$.
Since $P^{-1}=P^T$, we now have $A=PDP^T$.

Example 85.

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(b) Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$, and so A has eigenvalues $5, 0$.

The 5-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The 0-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Important observation. The 5-eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the 0-eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal!

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose P so that $P^{-1} = P^T$, which means that P must be **orthogonal** (meaning orthonormal columns). [Choosing such a P is only possible if the eigenspaces of A are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, we then have $A = PDP^T$.