The fundamental theorem

Review. The four **fundamental subspaces** associated with a matrix A are

$$\operatorname{col}(A),\quad \operatorname{row}(A),\quad \operatorname{null}(A),\quad \operatorname{null}(A^T).$$

Note that $\operatorname{row}(A) = \operatorname{col}(A^T)$. (In particular, we usually write vectors in $\operatorname{row}(A)$ as column vectors.)

Comment. $\operatorname{null}(A^T)$ is called the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\operatorname{null}(A) \iff A\boldsymbol{x} = \boldsymbol{0}$.

Likewise, $\boldsymbol{x}$ is in $\operatorname{null}(A^T) \iff A^T\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x}^T A = \boldsymbol{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

Review. The **rank** of a matrix is the number of pivots in its RREF.

Equivalently, as showcased in the next result, the rank is the dimension of either the column or the row space.

Theorem 35. **(Fundamental Theorem of Linear Algebra, Part I)**

Let A be an $m \times n$ matrix of **rank** r.

- $\dim \operatorname{col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \operatorname{row}(A) = r$ (subspace of $\mathbb{R}^n$) $\qquad \operatorname{row}(A) = \operatorname{col}(A^T)$
- $\dim \operatorname{null}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \operatorname{null}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Example 36. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Solution. Make sure that, for such a simple matrix, you can see all of these that at a glance!

$\operatorname{col}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, $\operatorname{null}(A) = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, $\operatorname{null}(A^T) = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$

Important observation. The basis vectors for $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are orthogonal! $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Always. Vectors in $\operatorname{null}(A)$ are orthogonal to vectors in $\operatorname{row}(A)$. In short, $\operatorname{null}(A)$ is orthogonal to $\operatorname{row}(A)$.

Why? Suppose that $\boldsymbol{x}$ is in $\operatorname{null}(A)$. That is, $A\boldsymbol{x} = \boldsymbol{0}$. But think about what $A\boldsymbol{x} = \boldsymbol{0}$ means (row-product rule). It means that the inner product of every row with $\boldsymbol{x}$ is zero. Which implies that $\boldsymbol{x}$ is orthogonal to the row space.

Theorem 37. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\operatorname{null}(A)$ is orthogonal to $\operatorname{row}(A)$. (both subspaces of $\mathbb{R}^n$)

Note that $\dim \operatorname{null}(A) + \dim \operatorname{row}(A) = n$. Hence, the two spaces are orthogonal complements.

- $\operatorname{null}(A^T)$ is orthogonal to $\operatorname{col}(A)$.

Again, the two spaces are orthogonal complements. (This is just the first part with A replaced by A^T.)

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Example 38. Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$. Check that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3-3R_1\Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \underset{\rightsquigarrow}{R_3-\frac{3}{2}R_2\Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\rightsquigarrow}{-\frac{1}{2}R_2\Rightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{R_1-R_2\Rightarrow R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

$\operatorname{null}(A)$ and $\operatorname{row}(A)$ are indeed orthogonal, as certified by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$$

In fact, $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements because the dimensions add up to $2+2=4$.

In particular, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ form a basis of all of $\mathbb{R}^4$.

Example 39. (extra) Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ (as well as $\operatorname{null}(A^T)$ and $\operatorname{col}(A)$) are orthogonal complements.

Partial solution. One can almost see that $\operatorname{rank}(A) = 3$. Hence, the dimensions of the fundamental subspaces are ...

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Consistency of a system of equations

Example 40. (warmup) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Note that this means that the system of equations $\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 1 \\ 3x_1 & + & x_2 & = & 1 \\ & & 5x_2 & = & 1 \end{array}$ can also be written as $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

[This was the motivation for introducing matrix-vector multiplication.]

In the same way, any system can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where A is a matrix and $\boldsymbol{b}$ a vector.
In particular, this makes it obvious that:

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is consistent} \quad \Longleftrightarrow \quad \boldsymbol{b} \text{ is in } \operatorname{col}(A)$$

Recall that, by the FTLA, $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are orthogonal complements.

Theorem 41. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\Longleftrightarrow$ $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$

Proof. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\Longleftrightarrow$ $\boldsymbol{b}$ is in $\operatorname{col}(A)$ $\overset{\text{FTLA}}{\Longleftrightarrow}$ $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$

Note. $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$ means that $\boldsymbol{y}^T\boldsymbol{b} = 0$ whenever $\boldsymbol{y}^T A = \boldsymbol{0}$. Why?!

Example 42. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

Solution. (old)

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 1 & b_2 \\ 0 & 5 & b_3 \end{array}\right] \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 5 & b_3 \end{array}\right] \overset{R_3 + R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -5 & -3b_1 + b_2 \\ 0 & 0 & -3b_1 + b_2 + b_3 \end{array}\right]$$

So, $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $-3b_1 + b_2 + b_3 = 0$.

Solution. (new) We determine a basis for $\operatorname{null}(A^T)$:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -5 & 5 \end{bmatrix} \overset{-\frac{1}{5}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \end{bmatrix} \overset{R_1 - 3R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

We read off from the RREF that $\operatorname{null}(A^T)$ has basis $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

$\boldsymbol{b}$ has to be orthogonal to $\operatorname{null}(A^T)$. That is, $\boldsymbol{b} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0$. As above!

Comment. Below is how we can use Sage to (try and) solve $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

```
Sage] A = matrix([[1,2],[3,1],[0,5]])
Sage] A.solve_right(vector([1,1,2]))
```

$$\left(\frac{1}{5}, \frac{2}{5}\right)$$

```
Sage] A.solve_right(vector([1,1,1]))
```

```
Traceback (most recent call last):
ValueError: matrix equation has no solutions
```

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