Review. Recall the Gauss–Jordan method of computing A^{-1}. Starting with the augmented matrix $[A \mid I]$, we do Gaussian elimination until we obtain the RREF, which will be of the form $[I \mid A^{-1}]$ so that we can read off A^{-1}.

Why does that work? By our discussion, the steps of Gaussian elimination can be expressed by multiplication (on the left) with a matrix B. Only looking at the first part of the augmented matrix, and since the RREF of an invertible matrix is I, we have $BA = I$, which means that we must have $B = A^{-1}$. The other part of the augmented matrix (which is I initially) gets multiplied with $B = A^{-1}$ as well, so that, in the end, it is $BI = A^{-1}$. That's why we can read off A^{-1}!

For instance. To invert $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ using the Gauss–Jordan method, we would proceed as follows:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -6 & 0 & 1 \end{array}\right] \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -8 & -2 & 1 \end{array}\right] \overset{\substack{\frac{1}{2}R_1 \Rightarrow R_1 \\ -\frac{1}{8}R_2 \Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array}\right] \overset{R_1 - \frac{1}{2}R_2 \Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{8} & \frac{1}{16} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array}\right]$$

We conclude that $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$.

Of course, for 2×2 matrices it is much simpler to use the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Review: Vector spaces, bases, dimension, null spaces

Review.

- Vectors are things that can be **added** and **scaled**.
- Hence, given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, the most general we can do is form the **linear combination** $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n$. The set of all these linear combinations is the **span** of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, denoted by $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.
- Vector **spaces** are spans.

  Equivalently. Vector spaces are sets of vectors so that the result of adding and scaling remains within that set.

  Homework. Of course, the latter is a very informal statement. Revisit the formal definition, probably consisting of a list of axioms, and observe how that matches with the above (for instance, several of the axioms are concerned with addition and scaling satisfying the "expected" rules).

- Recall that vectors from a vector space V form a **basis** of V if and only if
  - the vectors span V, and
  - the vectors are (linearly) independent.

  Equivalently. $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ from V form a basis of V if and only if every vector in V can be expressed as a unique linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

  Just checking. Make sure that you can define precisely what it means for vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ to be independent.

- The **dimension** of a vector space V is the number of vectors in a basis for V.

  No matter what basis one chooses for V, it always has the same number of vectors.

Example 12. $\mathbb{R}^3$ is the vector space of all vectors with 3 real entries.

$\mathbb{R}$ itself refers to the set of real numbers. We will later also discuss $\mathbb{C}$, the set of complex numbers.

The **standard basis** of $\mathbb{R}^3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The dimension of $\mathbb{R}^3$ is 3.

Review. The **null space** $\mathrm{null}(A)$ of a matrix A consists of those vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$.

Make sure that you see why $\mathrm{null}(A)$ is a vector space. [For instance, if you pick two vectors in $\mathrm{null}(A)$ why is it that the sum of them is in $\mathrm{null}(A)$ again?]

Example 13. What is $\mathrm{null}(A)$ if the matrix A is invertible?

Solution. If A is invertible, then $A\boldsymbol{x} = \mathbf{0}$ has the unique solution $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$.
Hence, $\mathrm{null}(A) = \{\mathbf{0}\}$ which is the trivial vector space (consisting of only the null vector) and has dimension 0.

Example 14. Compute a basis for $\mathrm{null}(A)$ where $A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$.

Solution. We perform row operations and obtain

$$\mathrm{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\substack{R_2+2R_1 \Rightarrow R_2 \\ R_3+R_1 \Rightarrow R_3}}{=} \mathrm{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right) \overset{\substack{-R_1 \Rightarrow R_1 \\ -\frac{1}{3}R_2 \Rightarrow R_2}}{=} \mathrm{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

From the RREF, we can now read off the general solution to $A\boldsymbol{x} = \mathbf{0}$:

- x_1 and x_2 are pivot variables. [For each we have an equation expressing it in terms of the other variables; for instance, $x_1 - 2x_3 = 0$ tells us that $x_1 = 2x_3$.]
- x_3 is a free variable. [There is no equation forcing a value on x_3.]
- Hence, without computation, we see that the general solution is $\begin{bmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$.

In other words, a basis is $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Comment. We are starting with the three equations $-x_1 + 2x_3 = 0$, $2x_1 - 3x_2 + 2x_3 = 0$, $x_1 - 2x_3 = 0$. Performing row operations on the matrix is the same as combining these equations (with the objective to form simpler equations by eliminating variables).

Example 15. Compute a basis for $\mathrm{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$.

Solution.

$$\mathrm{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\substack{R_2-R_1 \Rightarrow R_2 \\ R_3-\frac{1}{2}R_1 \Rightarrow R_3}}{=} \mathrm{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \overset{\frac{1}{2}R_1 \Rightarrow R_1}{=} \mathrm{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

This time, x_2 and x_3 are free variables. The general solution is $\begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.
Hence, a basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Armin Straub
straub@southalabama.edu