Example 7. Let us do Gaussian elimination on $A=\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ until we have an echelon form:

$$A=\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \overset{R_2-2R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

As last class, the row operation can be encoded by multiplication with an "almost identity matrix" E:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}}_{A}=\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_{U}$$

Since $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (no calculation needed; this is the row operation $R_2+2R_1\Rightarrow R_2$ which reverses our above operation), this means that

$$A=E^{-1}U=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}.$$

We factored A as the product of a lower and an upper triangular matrix!

$A=LU$ is known as the **LU decomposition** of A.

L is lower triangular, U is upper triangular.

If A is $m\times n$, then L is an invertible lower triangular $m\times m$ matrix, and U is a usual **echelon form** of A. Every matrix A has a LU decomposition (after possibly swapping some rows of A first).

- The matrix U is just the echelon form of A produced during Gaussian elimination.
- The matrix L can be constructed, entry-by-entry, by simply recording the row operations used during Gaussian elimination. (No extra work needed!)

Example 8. Determine the LU decomposition of $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution. $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{R_2-3R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ translates into $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (no calculation needed!), we therefore have $A=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Example 9. Determine the LU decomposition of $A=\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix}$.

Solution. We perform Gaussian elimination until we arrive at an echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \overset{\substack{R_2-3R_1\Rightarrow R_2 \\ R_3+2R_1\Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \overset{R_3+8R_2\Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

Observe that we can reverse both of these steps using the row operations $\begin{array}{c} R_2+3R_1\Rightarrow R_2 \\ R_3-2R_1-8R_2\Rightarrow R_3 \end{array}$.

Encoding these in L, the corresponding LU decomposition of A is

$$A=LU=\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 & 1 \\ & -1 & 1 & -1 \\ & & 9 & -5 \end{bmatrix}.$$

Note that no further computation was required to obtain L. (The entries in the matrix L are precisely the (negative) coefficients in the original row operations.)

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Comment. By contrast, combining the operations $\begin{array}{c} R_2 - 3R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3 \end{array}$ and $R_3 + 8R_2 \Rightarrow R_3$ requires computation. That is because we change R_2 in the first step, and then use the changed R_2 in the second step. Indeed, note that

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -3 & 1 & \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ -22 & 8 & 1 \end{bmatrix},$$

so the combined operations are $\begin{array}{c} R_2 - 3R_1 \Rightarrow R_2 \\ R_3 - 22R_1 + 8R_2 \Rightarrow R_3 \end{array}$ (you can also see that directly from the operations).

On the other hand, there was no such complication when combining the reversed operations:

Combining $R_3 - 8R_2 \Rightarrow R_3$ and $\begin{array}{c} R_2 + 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3 \end{array}$ simply results in $\begin{array}{c} R_2 + 3R_1 \Rightarrow R_2 \\ R_3 - 2R_1 - 8R_2 \Rightarrow R_3 \end{array}$, as used above.

The difference is that, here, we change R_3 in the first step but then don't use the changed R_3 in the second step. In terms of matrix multiplication, we have

$$\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & -8 & 1 \end{bmatrix},$$

where, because of their special form, the product of the two lower triangular matrices is just "putting together" the entries (unlike in the non-reversed product).

Review. The **RREF** (row-reduced echelon form) of A is obtained from an echelon form by

- scaling the pivots to 1, and then
- eliminating the entries above the pivots.

A typical RREF has the shape [$*$ represents an entry that could be anything]

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & * \\ & & 1 & * & * & 0 & * \\ & & & & & 1 & * \end{bmatrix}$$

Example 10. Let's compute the RREF of the 3×4 matrix from Example 9.

Solution.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \overset{\substack{R_2 - 3R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \overset{R_3 + 8R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

$$\overset{\substack{-R_2 \Rightarrow R_2 \\ \frac{1}{9}R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \overset{\substack{R_1 - 2R_3 \Rightarrow R_1 \\ R_2 + R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 0 & \frac{19}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix}$$

Example 11. The RREF of $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ from earlier is the 2×2 identity matrix.

Comment. That's not surprising: A square matrix is invertible if and only if its RREF is the identity matrix. If that isn't obvious to you, think about how you invert a matrix using Gaussian elimination (reviewed next).

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