Homework Set 11

Problem 1

Example 18. Determine the pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}$.

Solution. For such diagonal matrices, we only need to invert the diagonal entries and transpose the dimensions. $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/5 \\ 0 & 0 \end{bmatrix}$

Problem 2

Example 19. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$ (without computing the SVD first).

Solution. This matrix clearly has full column rank (because the two columns are not multiples of each other). Hence, $A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 13 & -6 \\ -6 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 8 & 12 & 39 \\ -27 & 26 & 18 \end{bmatrix}$.

Problem 3

Example 20. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$ (by computing the SVD first).

Solution. (skipping most work) As observed in Examples 176 and 177 in the lecture notes, we can avoid almost all computations and conclude that, if $A = \boldsymbol{a}^T$ is a row vector, then

$$A^+ = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Solution. (too much work but good pratice) Let us embrace the opportunity to practice. We first compute the SVD of A:

First, we need to diagonalize $A^T A = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}$. Let us write $|A|$ for $\det(A)$:

$$\begin{aligned} \begin{vmatrix} 4-\lambda & -4 & 2 \\ -4 & 4-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} &= (4-\lambda) \cdot \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - (-4) \cdot \begin{vmatrix} -4 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} -4 & 4-\lambda \\ 2 & -2 \end{vmatrix} \\ &= (4-\lambda) \cdot (\lambda^2 - 5\lambda) + 4 \cdot (4\lambda) + 2 \cdot (2\lambda) = -\lambda^3 + 9\lambda^2 = \lambda^2 (9-\lambda) \end{aligned}$$

Hence, the eigenvalues of $A^T A$ are $9, 0, 0$.

- $\lambda = 9$: $\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \overset{\substack{R_2 - \frac{4}{5}R_1 \Rightarrow R_2 \\ R_3 + \frac{2}{5}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & -\frac{18}{5} & -\frac{36}{5} \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix}$
  $\overset{\substack{-\frac{1}{5}R_1 \Rightarrow R_1 \\ -\frac{5}{9}R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - \frac{4}{5}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

  Hence, the 9-eigenspace has basis $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

- $\lambda=0$: $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{\substack{R_2+R_1\Rightarrow R_2 \\ R_3-\frac{1}{2}R_1\Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4}R_1\Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  Hence, the 0-eigenspace has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$. For the SVD we have to turn this basis into an orthogonal one.

  Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

Thus $A^T A = P D P^T$ with $D = \begin{bmatrix} 9 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

[We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V \Sigma^2 V^T$, we conclude that $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = 1$.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

Using the SVD of A, we can easily obtain its pseudoinverse:

$$A^+ = V\Sigma^+ U^T = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Comments. This was good practice computing SVDs but we did a lot of work that we could have simplified: Can you see why it was clear that $A^T A$ was going to have 0 as a repeated eigenvalue? Can you see why the last two columns of P are irrelevant in our computation? Can you see how we could have obtained the first column of P without computation? [Also, can you argue geometrically why the pseudoinverse is what it is?]

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Problem 4

Example 21. Find the smallest norm solution to $4x_1 + 3x_2 + 5x_3 = 3$.

Solution. If $A = [\ 4\ \ 3\ \ 5\ ]$, then the smallest norm solution is $\boldsymbol{x} = A^+[\ 3\ ]$.

From earlier computations (see, for instance, Example 177) we know that $A^+ = \frac{1}{4^2+3^2+5^2}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Hence, the smallest norm solution is $\boldsymbol{x} = A^+[\ 3\ ] = \frac{3}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Problem 5

Example 22. Determine the best rank 1 approximation of $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution. We first compute the SVD of A:

- First, we need to diagonalize $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\right) = (2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$

  Hence, the eigenvalues of A^TA are $6, 1$.

  - $\lambda = 6$: $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \overset{R_2 - \frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{4}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

    Hence, the 6-eigenspace has basis $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

  - $\lambda = 1$: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \overset{R_2 + 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

    Hence, the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 6 & \\ & 1 \end{bmatrix}$ and $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

  [We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2V^T$, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}$.

  For the rank 1 approximation, we only need the first column of U, so we stop here.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} -5/\sqrt{30} & * & * \\ -2/\sqrt{30} & * & * \\ -1/\sqrt{30} & * & * \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

From the SVD of A, we obtain the best rank 1 approximation by only using the first columns of U and V (and truncating Σ to a 1×1 matrix):

Thus, the best rank 1 approximation of A is $\frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}[\ \sqrt{6}\ ]\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}^T = \sqrt{\frac{6}{30 \cdot 5}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}[\ -1\ \ 2\ ] = \frac{1}{5}\begin{bmatrix} 5 & -10 \\ 2 & -4 \\ 1 & -2 \end{bmatrix}$.

Comment. Like for U, we could have omitted the computation of the 1-eigenvector (second column of V).

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