Homework Set 8

Problem 1

You find the answer to this problem at the very beginning of Lecture 22.

Problem 2

Example 1. Let A be the matrix for reflecting through the plane spanned by $\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

Diagonalize A as $A = PDP^{-1}$.

Solution. The eigenvalues of A are $1, 1, -1$. The 1-eigenspace of A is W, and the -1-eigenspace is $W^\perp$.

If $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$, then $W^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 4 & -2 \\ 0 & 2 & -3 \end{bmatrix} \right)$.

It follows from $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 2 & -3 \end{bmatrix} \overset{\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \overset{R_1 - 4R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$ that $W^\perp = \operatorname{span}\left\{ \begin{bmatrix} -4 \\ 3/2 \\ 1 \end{bmatrix} \right\}$.

We can therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & -4 \\ 4 & 2 & 3/2 \\ -2 & -3 & 1 \end{bmatrix}$.

Comment. From here, we can produce a diagonalization of the form $A = PDP^T$. How?

Problem 3

Example 2. Solve the initial value problem $y' = 3y$ with $y(0) = 8$.

Solution. The general solution to $y' = 3y$ is $y(t) = Ce^{3t}$.
Since $y(0) = Ce^0 = C = 8$, we conclude that the unique solution to the IVP is $y(t) = 8e^{3t}$.

Problem 4

Example 3. Compute $\exp(Dt)$ for the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Solution. $e^{Dt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$

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Problem 5

Example 4. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 11 & 8 \\ -12 & -9 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- First, we diagonalize $A = \begin{bmatrix} 11 & 8 \\ -12 & -9 \end{bmatrix}$:
  $\det\left(\begin{bmatrix} 11-\lambda & 8 \\ -12 & -9-\lambda \end{bmatrix}\right) = (11-\lambda)(-9-\lambda) + 96 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$
  Hence, the eigenvalues of A are $-1, 3$.
  - $\lambda = -1$: $\begin{bmatrix} 12 & 8 \\ -12 & -8 \end{bmatrix} \overset{R_2+R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 12 & 8 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{12}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$
    Hence, the -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 12 & 8 \\ -12 & -8 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$.
  - $\lambda = 3$: The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} 8 & 8 \\ -12 & -12 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  In conclusion, $A = PDP^{-1}$ with $P = \begin{bmatrix} -2/3 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 3 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} -\frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{3t} \end{bmatrix}}_{\begin{bmatrix} -\frac{2}{3}e^{-t} & -e^{3t} \\ e^{-t} & e^{3t} \end{bmatrix}} \underbrace{\frac{1}{-\frac{2}{3}+1}\begin{bmatrix} 1 & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 9 \\ -9 \end{bmatrix}} = \begin{bmatrix} -6e^{-t} + 9e^{3t} \\ 9e^{-t} - 9e^{3t} \end{bmatrix} \end{aligned}$$

Problem 6

Example 5. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} -5 & 0 & -3 \\ -6 & -2 & -6 \\ 6 & 0 & 4 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

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Solution.

- First, we diagonalize $A = \begin{bmatrix} -5 & 0 & -3 \\ -6 & -2 & -6 \\ 6 & 0 & 4 \end{bmatrix}$:

$$\underbrace{\det\left(\begin{bmatrix} -5-\lambda & 0 & -3 \\ -6 & -2-\lambda & -6 \\ 6 & 0 & 4-\lambda \end{bmatrix}\right)}_{\text{expand by 2nd column}} = (-2-\lambda)\det\left(\begin{bmatrix} -5-\lambda & -3 \\ 6 & 4-\lambda \end{bmatrix}\right)$$
$$= (-2-\lambda)((-5-\lambda)(4-\lambda)+18) = (-2-\lambda)(\lambda^2+\lambda-2)$$
$$= (-2-\lambda)(\lambda-1)(\lambda+2)$$

Hence, the eigenvalues of A are $1, -2, -2$.

  - $\lambda = 1$: $\begin{bmatrix} -6 & 0 & -3 \\ -6 & -3 & -6 \\ 6 & 0 & 3 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2 - R_1 \Rightarrow R_2}{R_3 + R_1 \Rightarrow R_3}} \begin{bmatrix} -6 & 0 & -3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{\overset{-\frac{1}{6}R_1 \Rightarrow R_1}{-\frac{1}{3}R_2 \Rightarrow R_2}} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

    Hence, the -1-eigenspace has basis $\begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$.

  - $\lambda = -2$: $\begin{bmatrix} -3 & 0 & -3 \\ -6 & 0 & -6 \\ 6 & 0 & 6 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2 - 2R_1 \Rightarrow R_2}{R_3 + 2R_1 \Rightarrow R_3}} \begin{bmatrix} -3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{-\frac{1}{3}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

    Since, $\begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, the -2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

In conclusion, $A = PDP^{-1}$ with $P = \begin{bmatrix} -1/2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & -2 & \\ & & -2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = Pe^{Dt}P^{-1}\boldsymbol{y}_0$$
$$= \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{-2t} & \\ & & e^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Note that $\boldsymbol{x} = \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is equivalent to $\begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. We solve this to find $\boldsymbol{x}$:

It follows from $\left[\begin{array}{ccc|c} -\frac{1}{2} & 0 & -1 & 3 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2 - 2R_1 \Rightarrow R_2}{R_3 + 2R_1 \Rightarrow R_3}} \left[\begin{array}{ccc|c} -\frac{1}{2} & 0 & -1 & 3 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & -1 & 7 \end{array}\right] \underset{\rightsquigarrow}{\overset{-2R_1 \Rightarrow R_1}{-R_3 \Rightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -6 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -7 \end{array}\right]$

$\underset{\rightsquigarrow}{\overset{R_1 - 2R_3 \Rightarrow R_1}{R_2 - 2R_3 \Rightarrow R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -7 \end{array}\right]$ that $\boldsymbol{x} = \begin{bmatrix} 8 \\ 8 \\ -7 \end{bmatrix}$. Therefore:

$$\boldsymbol{y}(t) = \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{-2t} & \\ & & e^{-2t} \end{bmatrix} \begin{bmatrix} 8 \\ 8 \\ -7 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8e^t \\ 8e^{-2t} \\ -7e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^t + 7e^{-2t} \\ -8e^t + 8e^{-2t} \\ 8e^t - 7e^{-2t} \end{bmatrix}$$

Problem 7

Example 6. Given $e^{Mt} = \begin{bmatrix} -2e^{-t} + 3e^{3t} & -e^{-t} + e^{3t} \\ 6e^{-t} - 6e^{3t} & 3e^{-t} - 2e^{3t} \end{bmatrix}$, determine the matrix M.

Solution. (via matrix powers) Without computations, we can conclude that (by replacing $e^{\lambda t}$ with λ^n)

$$M^n = \begin{bmatrix} -2(-1)^n + 3 \cdot 3^n & -(-1)^n + 3^n \\ 6(-1)^n - 6 \cdot 3^n & 3(-1)^n - 2 \cdot 3^n \end{bmatrix}.$$

In particular, setting $n = 1$, we find $M = \begin{bmatrix} 2 + 3 \cdot 3 & 1 + 3 \\ -6 - 6 \cdot 3 & -3 - 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -24 & -9 \end{bmatrix}$.

Solution. (via derivative) Alternatively, we can find M by computing $\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = M e^{Mt}$ and then setting $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{Mt} = \begin{bmatrix} 2e^{-t} + 9e^{3t} & e^{-t} + 3e^{3t} \\ -6e^{-t} - 18e^{3t} & -3e^{-t} - 6e^{3t} \end{bmatrix}.$$

In particular, now setting $t = 0$, we find $M = \begin{bmatrix} 2 + 9 & 1 + 3 \\ -6 - 18 & -3 - 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -24 & -9 \end{bmatrix}$.

Problem 8

Example 7. Write the (third-order) differential equation $y''' = 5y'' + 6y' + 2y$ as a system of (first-order) differential equations.

Solution. Write $\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$. Then $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 6 & 5 \end{bmatrix} \boldsymbol{y}$.

[Note how the third row of the matrix encodes $y''' = 2y + 6y' + 5y''$, while the first and second row encode the (trivial) equations $y' = y'$ and $y'' = y''$.].

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