Midterm #2

MATH 316 — Linear Algebra II
Friday, Mar 31, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$ has characteristic polynomial $(1-\lambda)(5-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$.

  Hence, the eigenvalues of A are $2, 4$.

  The 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} e^{4t} & 3e^{2t} \\ e^{4t} & e^{2t} \end{bmatrix}} \underbrace{\left(-\frac{1}{2}\right) \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 3e^{2t} - e^{4t} \\ e^{2t} - e^{4t} \end{bmatrix} \end{aligned}$$

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Problem 2. (1+4+1 points) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) The next two terms are $a_2 = \boxed{}$ and $a_3 = \boxed{}$.

(b) A Binet-like formula for a_n is $a_n = \boxed{}$, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \boxed{}$.

Solution.

(a) $a_2 = 10$, $a_3 = 26$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ are $2, -1$.

Hence, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = 2\alpha_1 - \alpha_2 = 8$.

Solving, we find $\alpha_1 = 3$ and $\alpha_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$.

Problem 3. (2 points) Let A be the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Determine an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Solution. The normal direction is spanned by $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

Normalizing all vectors, we can choose $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{10} & -3/\sqrt{10} \\ 0 & 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Problem 4. (1+1+2 points) Fill in the blanks.

(a) An example of a 2×2 matrix with eigenvalue $\lambda = 5$ that is not diagonalizable is $\boxed{}$.

(b) If $N^3 = \mathbf{0}$, then $e^{Nt} = \boxed{}$.

(c) How many different Jordan normal forms are there in the following cases?

- A 4×4 matrix with eigenvalues $2, 5, 5, 5$? $\boxed{}$
- A 8×8 matrix with eigenvalues $1, 1, 2, 2, 4, 4, 4, 4$? $\boxed{}$

Solution.

(a) An example of a 2×2 matrix with eigenvalue $\lambda = 5$ that is not diagonalizable is $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. (This is a Jordan block!)

(b) If $N^3 = \mathbf{0}$, then $e^{Nt} = I + Nt + \frac{1}{2}N^2t^2$.

(c) $1 \cdot 3 = 3$ and $2 \cdot 2 \cdot 5 = 20$ different Jordan normal forms.

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Problem 5. (4 points) Fill in the blanks.

(a) Let A be the 4×4 matrix for orthogonally projecting onto a 2-dimensional subspace of $\mathbb{R}^4$.

Then $\det(A) =$ ____ , and the eigenvalues (indicate if repeated) of A are ____ .

(b) If A is a projection matrix, then $A^{2024} =$ ____ . If B is a reflection matrix, then $B^{2024} =$ ____ .

(c) If A has eigenvalue 2, then A^3 has eigenvalue ____ , $3A$ eigenvalue ____ , and A^T eigenvalue ____ .

(d) If $A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, then $A^n =$ ____ and $e^{At} =$ ____ .

Solution.

(a) $\det(A) = 0$, and the eigenvalues of A are $0, 0, 1, 1$.

(b) If A is a projection matrix, then $A^{2024} = A$. (Because $A^2 = A$.)

If B is a reflection matrix, then $B^{2024} = I$. (Because $B^2 = I$.)

(c) If A has eigenvalue 2, then A^3 has eigenvalue $2^3 = 8$, $3A$ eigenvalue $3 \cdot 2 = 6$, and A^T eigenvalue 2.

(d) If $A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$, then $A^n = \begin{bmatrix} (-2)^n & \\ & 4^n \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{-2t} & \\ & e^{4t} \end{bmatrix}$.

Problem 6. (2 points) Convert the third-order differential equation

$$y''' = 6y'' - 3y' - 10y, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

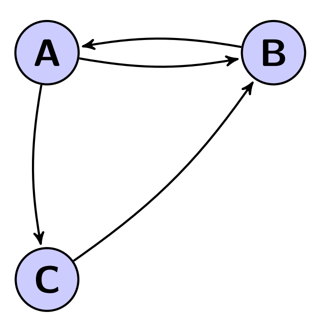
to a system of first-order differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 6y'' - 3y' - 10y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -10y_1 - 3y_2 + 6y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Problem 7. (4 points) Suppose the internet consists of only the three webpages A, B, C which link to each other as indicated in the diagram.



Rank these webpages by computing their PageRank vector.

PageRank vector: ____ . Ranking of websites: ____ .

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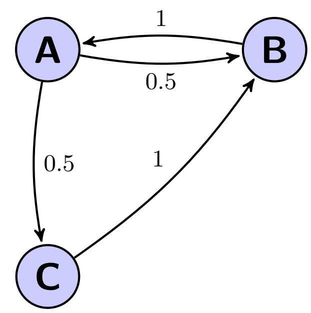
Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.



We obtain the following transition behaviour:

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. This is the PageRank vector.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

We therefore rank A and B highest (tied), and C lowest.

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(extra scratch paper)