Example 127. We only discuss linear differential equations (DEs). Non-linear DEs include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$.

The order of a DE indicates the highest occuring derivative.

Note, however, that $y'' = \sin(t)y' + y$ is a linear DE, because y and its derivatives occur linearly.

We will see here how to solve those linear DEs which have constant coefficients. That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Review.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

- If we have the diagonalization $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$ (and $e^{At} = Pe^{Dt}P^{-1}$).

- If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 128. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(1-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

  Hence, the eigenvalues of A are $-1, 2$.

  The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} 2e^{-t} & -e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2e^{-t} + e^{2t} \\ e^{-t} - e^{2t} \end{bmatrix} \end{aligned}$$

Check. Since it is simple to check, it would be almost criminal to not verify that $\boldsymbol{y}(0) = \begin{bmatrix} 2+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Example 129. (homework) Suppose that $e^{Mt} = \frac{1}{10}\begin{bmatrix} e^t + 9e^{2t} & 3e^t - 3e^{2t} \\ 3e^t - 3e^{2t} & 9e^t + e^{2t} \end{bmatrix}$.

(a) Without doing any computations, determine M^n.

(b) What is M?

(c) Without doing any computations, determine the eigenvalues and eigenvectors of M.

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Solution.

(a) Recall that $e^{Mt} = Pe^{Dt}P^{-1}$ while $M^n = PD^nP^{-1}$, provided that $M = PDP^{-1}$. The fact the formula for e^{Mt} features e^t and e^{2t}, means that the eigenvalues of M must be 1 and 2. Hence,

$$D = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix}, \quad D^n = \begin{bmatrix} 1 & \\ & 2^n \end{bmatrix}.$$

Therefore, we just need to replace e^t by $1^n = 1$ as well as e^{2t} by 2^n to get:

$$M^n = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2^n & 3 - 3 \cdot 2^n \\ 3 - 3 \cdot 2^n & 9 + 2^n \end{bmatrix}$$

(b) In particular, we see that the underlying matrix is $M = M^1 = \frac{1}{10}\begin{bmatrix} 1 + 9 \cdot 2 & 3 - 3 \cdot 2 \\ 3 - 3 \cdot 2 & 9 + 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$.

[Alternatively, we can find M by computing $\frac{\mathrm{d}}{\mathrm{d}t}e^{Mt} = Me^{Mt}$ and then setting $t = 0$.]

(c) The eigenvalues are 1 and 2.

Looking at the coefficients of e^t in the first column of e^{Mt}, we can see that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 1-eigenvector.

[We can also look the second column of e^{Mt}, to obtain $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ which is a multiple and thus equivalent.]

Likewise, we find that $\begin{bmatrix} 9 \\ -3 \end{bmatrix}$ or, equivalently, $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

Higher-order differential equations

Example 130. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Note. The "trick" of looking at the pair $\begin{bmatrix} y \\ y' \end{bmatrix}$ instead of a single function is what we used to translate the Fibonacci recurrence into a 2×2 system.

Example 131. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix}\boldsymbol{y}$.