Review. Fibonacci numbers, Binet formula

Example 104. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -1$, $a_1 = 5$.

(a) Determine the first few terms of the sequence.

(b) Write down a matrix-vector version of the recursion.

(c) Find a Binet-like formula for a_n.

(d) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $-1, 5, 7, 29, 79, 245, 727, 2189, 6559, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

(c) **(solution using matrix powers)** Thus, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

After some work (do it!), we diagonalize $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$ with $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}}_{= \begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3^{n+1} - 2(-1)^{n+1} \\ 3^n - 2(-1)^n \end{bmatrix}$.

In particular, $a_n = 3^n - 2(-1)^n$.

(simplified solution) The eigenvalues of $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ are 3 and -1.

Looking back at our work above, we can see that a_n therefore must have a formula of the form $a_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n$ for some unknown constants C_1, C_2 which we still need to figure out

Using the two initial conditions, we get two equations:

($a_0 =$) $C_1 + C_2 = -1$, ($a_1 =$) $3C_1 - C_2 = 5$.

Solving, we find $C_1 = 1$ and $C_2 = -2$ so that, in conclusion, $a_n = 3^n - 2(-1)^n$.

(d) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_1 = 0$.

Definition 105. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called ***C*-finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $r_1 = r_2 = 1$.

In matrix-vector form. $$\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$$

Armin Straub
straub@southalabama.edu

By the same reasoning as for Fibonacci numbers, C-finite sequences have a Binet-like formula:

Theorem 106. (generalized Binet formula) Suppose the recursion matrix M has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n$$

for certain numbers $C_1, \ldots, C_d$.

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$.

Comment. A little more care is needed in the case that eigenvalues are repeated.

Corollary 107. Under the assumptions of the previous theorem, if λ_1 is the eigenvalue with the largest absolute value and $\lambda_1 > 0$, as well as $\alpha_1 \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1$.

Proof. This follows from $a_n = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n$ because, for large n, the term $C_1\lambda_1$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1} + \ldots + C_d\lambda_d^{n+1}}{C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n} = \frac{C_1\lambda_1 + C_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\lambda_d\left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\left(\frac{\lambda_d}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{C_1\lambda_1}{C_1} = \lambda_1.$$

□

Example 108. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, \ldots$
Note that this sequence is C-finite of order 3.

(b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

Expanding by the 2nd row: $\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$

The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$
[You will not be asked to find roots of cubic polynomials by hand.]
Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.
Using the three initial conditions, we get three equations:
(a_0=) $C_1 + C_2 + C_3 = 0$, ($a_1 =$) $-C_1 + 2C_2 + 3C_3 = -2$, ($a_2 =$) $C_1 + 4C_2 + 9C_3 = 2$.
Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.
Comment. Do you see how we might have found the characteristic polynomial directly from the recursion?

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).
Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 109. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.