Example 175. Find the best approximation of $f(x)=\sqrt{x}$ on the interval $[0,1]$ using a function of the form $y=a+bx$.

Important observation. The orthogonal projection of $f:[0,1]\to\mathbb{R}$ onto $\text{span}\{1,x\}$ is not simply the projection onto 1 plus the projection onto x. That's because 1 and x are not orthogonal:

$$\langle 1,x\rangle=\int_0^1 t\,\mathrm{d}t=\frac{1}{2}\neq 0.$$

Solution. To find an orthogonal basis for $\text{span}\{1,x\}$, following Gram–Schmidt, we compute

$$x-\begin{pmatrix}\text{projection of}\\ x\text{ onto }1\end{pmatrix}=x-\frac{\langle x,1\rangle}{\langle 1,1\rangle}1=x-\frac{1}{2}.$$

Hence, $1,x-\frac{1}{2}$ is an orthogonal basis for $\text{span}\{1,x\}$.
The orthogonal projection of $\sqrt{x}$ on $[0,1]$ onto $\text{span}\{1,x\}=\text{span}\left\{1,x-\frac{1}{2}\right\}$ therefore is

$$\frac{\langle\sqrt{x},1\rangle}{\langle 1,1\rangle}1+\frac{\left\langle\sqrt{x},x-\frac{1}{2}\right\rangle}{\left\langle x-\frac{1}{2},x-\frac{1}{2}\right\rangle}\left(x-\frac{1}{2}\right)=\frac{\int_0^1\sqrt{t}\,\mathrm{d}t}{\int_0^1 1\,\mathrm{d}t}+\frac{\int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t}{\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t}\left(x-\frac{1}{2}\right).$$

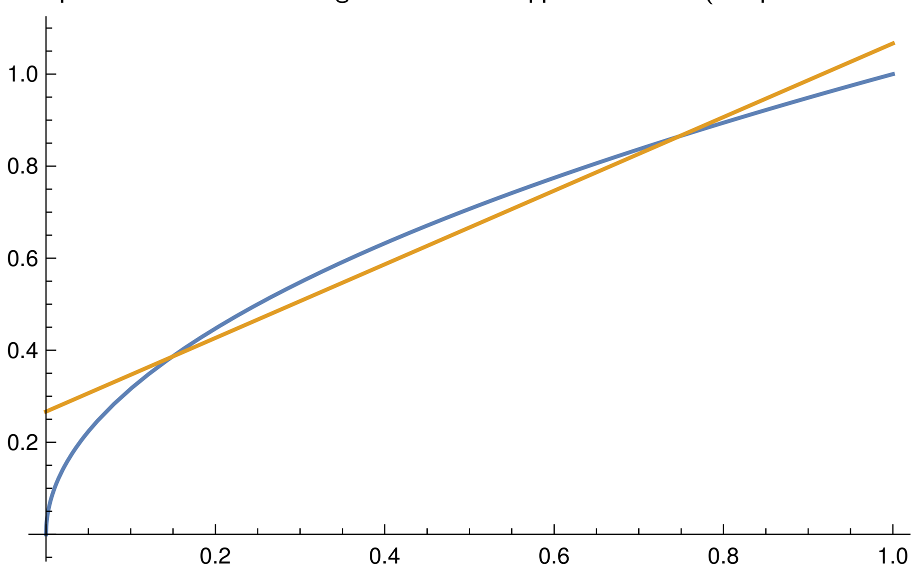
We compute the three new integrals:

$$\begin{aligned}\int_0^1\sqrt{t}\,\mathrm{d}t &= \left[\frac{2}{3}t^{3/2}\right]_0^1=\frac{2}{3}\\ \int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t &= \int_0^1\left(t^{3/2}-\frac{1}{2}t^{1/2}\right)\mathrm{d}t=\left[\frac{2}{5}t^{5/2}-\frac{1}{3}t^{3/2}\right]_0^1=\frac{2}{5}-\frac{1}{3}=\frac{1}{15}\\ \int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t &= \int_0^1\left(t^2-t+\frac{1}{4}\right)\mathrm{d}t=\left[\frac{1}{3}t^3-\frac{1}{2}t^2+\frac{1}{4}t\right]_0^1=\frac{1}{3}-\frac{1}{2}+\frac{1}{4}=\frac{1}{12}\end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_0^1\sqrt{t}\,\mathrm{d}t}{\int_0^1 1\,\mathrm{d}t}+\frac{\int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t}{\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t}\left(x-\frac{1}{2}\right)=\frac{2}{3}+\frac{12}{15}\left(x-\frac{1}{2}\right)=\frac{4}{5}x+\frac{4}{15}$$

The plot below confirms how good this linear approximation is (compare with the previous example):



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Example 176. Give a basis for the space of all polynomials.

Solution. $1, x, x^2, x^3, \ldots$

Indeed, every polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ can be written uniquely as a sum of these basis elements. ("can be" = span; "uniquely" = independent)

Comment. The dimension is ∞. But we can make a list of basis elements, which is the "smallest kind of ∞" and is referred to as **countably infinite**. For the space of all functions, no such list can be made.

Just for fun. Let us indicate this difference in infiniteness in a slightly simpler situation: first, the natural numbers $0, 1, 2, 3\ldots$ are infinite but they are countable, because we can make a (infinite but complete) list starting with a first, then a second element and so on (hence, the name "countable"). On the other hand, consider the real numbers between 0 and 1. Clearly, there are infinitely many such numbers. The somewhat shocking fact (first realized by Georg Cantor in 1874) is that every attempt of making a complete list of these numbers must fail because every list will inevitably miss some numbers. Here's a brief indication of how the famous diagonal argument goes: suppose you can make a list, say:

$$\begin{array}{ll} \#1 & 0.111111\ldots \\ \#2 & 0.123456\ldots \\ \#3 & 0.750000\ldots \\ \vdots & \end{array}$$

Now, we are going to construct a new number $x = 0.x_1 x_2 x_3 \ldots$ with decimal digits x_i in such a way that the digit x_i differs (by more than 1) from the ith digit of number $\#i$ on our list. For instance, $0.352\ldots$ in our case (for instance, $x_3 = 2$ differs from 0, the 3rd digit of sequence $\#3$). By construction, the number x is missing from the list.

Comment on fun. The statement "some infinities are bigger than others" nicely captures our observation. It appears in the book *The Fault in Our Stars* by John Green, where it is said by a cranky old author who attributes it to Cantor. Hazel, the main character, later reflects on that statement and compares $[0, 1]$ to $[0, 2]$. Can you explain why that is actually not what Cantor meant...?

Orthogonal polynomials

Let us think about the space of all polynomials (with real coefficients). On that space, we consider the dot product

$$\langle p_1, p_2 \rangle = \int_{-1}^{1} p_1(t) p_2(t) \mathrm{d}t. \qquad (1)$$

Comment. That dot product is useful if we are thinking about the polynomials as functions on $[-1, 1]$ only. You can, of course, consider any other interval and you will obtain a shifted version of what we get here.

Example 177. Are $1, x, x^2, \ldots$ orthogonal (with respect to the inner product (1))?

Solution. Since $\langle x^r, x^s \rangle = \int_{-1}^{1} t^r t^s \mathrm{d}t = \int_{-1}^{1} t^{r+s} \mathrm{d}t$, we find that $\langle x^r, x^s \rangle = \begin{cases} \frac{2}{r+s+1}, & \text{if } r+s \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$

Hence, if $r + s$ is odd, then the monomials x^r and x^s are orthogonal. On the other hand, if $r + s$ is even, then x^r and x^s are not orthogonal.

Example 178. Use Gram-Schmidt to produce an orthogonal basis $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$ for the space of polynomials with the dot product (1). Compute $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4$.

Instead of normalizing these polynomials, **standardize** them so that $\boldsymbol{p}_n(1) = 1$.

Solution. We construct an orthogonal basis $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$ from $1, x, x^2, \ldots$ as follows:

- Starting with 1, we find $\boldsymbol{p}_0(x) = 1$.

  For future reference, let us note that $\|\boldsymbol{p}_0\|^2 = \int_{-1}^{1} 1 \mathrm{d}x = 2$.

- Starting with x, Gram–Schmidt produces $x - \begin{pmatrix} \text{projection of} \\ x \text{ onto } \boldsymbol{p}_0 \end{pmatrix} = x - \frac{\langle x, \boldsymbol{p}_0 \rangle}{\langle \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle} \boldsymbol{p}_0 = x - \int_{-1}^{1} t \mathrm{d}t = x$.

  Again, that's already standardized, so that $\boldsymbol{p}_1(x) = x$.

  Comment. The previous problem already told us that x is orthogonal to 1.

  For future reference, let us note that $\|\boldsymbol{p}_1\|^2 = \int_{-1}^{1} t^2 \mathrm{d}t = \frac{2}{3}$.

- Starting with x, Gram–Schmidt produces $x^2 - \begin{pmatrix} \text{projection of } x^2 \\ \text{onto span}\{\boldsymbol{p}_0, \boldsymbol{p}_1\} \end{pmatrix} = x^2 - \frac{\langle x^2, \boldsymbol{p}_0 \rangle}{\langle \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle} \boldsymbol{p}_0 - \frac{\langle x^2, \boldsymbol{p}_1 \rangle}{\langle \boldsymbol{p}_1, \boldsymbol{p}_1 \rangle} \boldsymbol{p}_1$
  $= x^2 - \frac{1}{2} \int_{-1}^{1} t^2 \mathrm{d}t - \frac{x}{2/3} \int_{-1}^{1} t^3 \mathrm{d}t = x^2 - \frac{1}{3}$.

  Hence, standardizing, $\boldsymbol{p}_2(x) = \frac{1}{2}(3x^2 - 1)$.

  Comment. The previous problem told us that x^2 is orthogonal to x (but not to 1).

- Continuing, we find $\boldsymbol{p}_3(x) = \frac{1}{2}(5x^3 - 3x)$ and $\boldsymbol{p}_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

Comment. These famous polynomials are known as the **Legendre polynomials**. The Legendre polynomial $\boldsymbol{p}_n$ is an even function if n is even, and an odd function if n is odd (can you explain why?!).

An explicit formula is $\boldsymbol{p}_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k}$.

For instance, $\boldsymbol{p}_2(x) = \frac{1}{4}((x-1)^2 + 2^2(x-1)(x+1) + (x+1)^2) = \frac{1}{2}(3x^2 - 1)$.

https://en.wikipedia.org/wiki/Legendre_polynomials

Comment. Legendre polynomials are an example of **orthogonal polynomials**. Each choice of dot product gives rise to a family of such orthogonal polynomials.

https://en.wikipedia.org/wiki/Orthogonal_polynomials

Comment. It is also particularly natural to consider the dot product (1), where the integral is from 0 to 1. In that case, we obtain what's known as the shifted Legendre polynomials $\tilde{\boldsymbol{p}}_n(x) = \boldsymbol{p}_n(2x - 1)$.

Comment on other norms. Our choice of inner product

$$\langle f, g \rangle = \int_a^b f(t) g(t) \mathrm{d}t$$

for (square-integrable) functions on $[a, b]$ gives rise to the norm $\|f\| = \left(\int_a^b f(t)^2 \mathrm{d}t\right)^{1/2}$. This is known as the L^2-norm (and often written as $\|f\|_2$).

It is the continuous analog of the usual Euclidean norm $\|\boldsymbol{v}\| = (v_1^2 + v_2^2 + \ldots)^{1/2}$ (known as ℓ^2-norm).

There do exist other norms to measure the magnitude of vectors, such as the ℓ_1-norm $\|\boldsymbol{v}\|_1 = |v_1| + |v_2| + \ldots$ or, more generally, for $p \geqslant 1$, the ℓ_p-norms $\|\boldsymbol{v}\|_p = (|v_1|^p + |v_2|^p + \ldots)^{1/p}$.

Likewise, for functions, we have the L^p-norms $\|f\|_p = \left(\int_a^b f(t)^p \mathrm{d}t\right)^{1/p}$.

Only in the case $p = 2$ do these norms come from an inner product. That's a mathematical (as opposed to geometric) reason why we especially care about that case.

Linear transformations

Throughout, V and W are vector spaces.

Just like we went from column vectors to abstract vectors (such as polynomials), the concept of a matrix leads to abstract linear transformations.

In the other direction, picking a basis, abstract vectors can be represented as column vectors (see Lecture 35). Correspondingly, linear transformations can then be represented as matrices.

Definition 179. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It necessarily sends the zero vector in V to the zero vector in W:

- $T(\mathbf{0}) = \mathbf{0}$ [because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$]

Comment. Linear transformations are special functions and, hence, can be composed. For instance, if $T: V \to W$ and $S: U \to V$ are linear transformations, then $T \circ S$ is a linear transformation $U \to W$ (sending $\boldsymbol{x}$ to $T(S(\boldsymbol{x}))$). If S, T are represented by matrices A, B, then $T \circ S$ is represented by the matrix BA. In other words, matrix multiplication arises as the composition of (linear) functions.

Example 180. The **derivative** you know from Calculus I is linear.

Indeed, the map $D: \left\{ \begin{array}{c} \text{space of all} \\ \text{differentiable} \\ \text{functions} \end{array} \right\} \to \left\{ \begin{array}{c} \text{space of all} \\ \text{functions} \end{array} \right\}$ defined by $f(x) \mapsto f'(x)$ is a linear transformation:

- $\underbrace{D(f(x) + g(x))}_{(f(x)+g(x))'} = \underbrace{D(f(x))}_{f'(x)} + \underbrace{D(g(x))}_{g'(x)}$
- $\underbrace{D(cf(x))}_{(cf(x))'} = \underbrace{cD(f(x))}_{cf'(x)}$

These are among the first properties you learned about the derivative.

Similarly, the **integral** you love from Calculus II is linear:

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x, \quad \int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$$

In this form, we are looking at a map $T: \left\{ \begin{array}{c} \text{space of all} \\ \text{continuous} \\ \text{functions} \end{array} \right\} \to \mathbb{R}$ defined by $T(f(x)) = \int_a^b f(x)\mathrm{d}x$.

Example 181. Consider the space V of all polynomials $p(x)$ of degree 3 or less. The map D: $V \to V$ given by $p(x) \mapsto p'(x)$ is a linear. Write down the matrix M for this linear map with respect to the basis $1, x, x^2, x^3$.

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Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = 2x$.

Example 182. Consider the map

$$D: \left\{ \begin{array}{c} \text{space of poly's} \\ \text{of degree} \leqslant 3 \end{array} \right\} \to \left\{ \begin{array}{c} \text{space of poly's} \\ \text{of degree} \leqslant 2 \end{array} \right\}, \quad p(x) \mapsto p'(x).$$

Write down the matrix M for this linear map with respect to the bases $1, x, x^2, x^3$ and $1, x, x^2$.

Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 = 2x$.

Example 183. What is the pseudo-inverse of the matrix M from the previous example. Interpret your finding.

Solution. (final answer only) The pseudo-inverse of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

The corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$ and x^2 to $\frac{1}{3}x^3$. That is, the pseudo-inverse computes the antiderivative of each monomial.

Comment. This is not surprising, since we are familiar from Calculus with the concepts of derivatives and antiderivatives (or integrals), and that these are "pseudo" inverse to each other.

Comment. Similarly, the pseudo-inverse of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$.

Now, the corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$, x^2 to $\frac{1}{3}x^3$, and x^3 to 0. That is, the pseudo-inverse computes the antiderivative of each monomial, with the exception of x^3 which gets send to 0 (its antiderivative does not live in the space of polynomials of degree 3).

Example 184. (The April Fools' Day "proof" that $\pi = 4$, cont'd)

In that "proof", we are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm introduced as we did for functions.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as you learned in Calculus II, the arc length of a function $y = f(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f'(x)^2}\mathrm{d}x.$$

Observe that this involves f'. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x \right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

Fourier series

A **Fourier series** for a function $f(x)$ is a series of the form

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

You may have seen Fourier series in other classes before. Our goal here is to tie them in with what we have learned about orthogonality.

In these other classes, you would have seen formulas for the coefficients a_k and b_k. We will see where those come from.

Observe that the right-hand side combination of cosines and sines is 2π-periodic.

Let us consider (nice) functions on $[0, 2\pi]$.

Or, equivalently, functions that are 2π-periodic.

We know that a natural inner product for that space of functions is

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

Example 185. Show that $\cos(x)$ and $\sin(x)$ are orthogonal (in that sense).

Solution. $\langle\cos(x), \sin(x)\rangle = \int_0^{2\pi}\cos(t)\sin(t)\mathrm{d}t = \left[\frac{1}{2}(\sin(t))^2\right]_0^{2\pi} = 0$

In fact:

All the functions $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$ are orthogonal to each other!

Moreover, they form a basis in the sense that every other (nice) function can be written as a (infinite) linear combination of these basis functions.

Example 186. What is the norm of $\cos(x)$?

Solution. $\langle\cos(x), \cos(x)\rangle = \int_0^{2\pi}\cos(t)\cos(t)\mathrm{d}t = \pi$

Why? There's many ways to evaluate this integral. For instance:

- integration by parts
- using a trig identity
- here's a simple way:
  - $\int_0^{2\pi}\cos^2(t)\mathrm{d}t = \int_0^{2\pi}\sin^2(t)\mathrm{d}t$ ($\cos$ and $\sin$ are just a shift apart)
  - $\cos^2(t) + \sin^2(t) = 1$
  - So: $\int_0^{2\pi}\cos^2(t)\mathrm{d}t = \frac{1}{2}\int_0^{2\pi}1\,\mathrm{d}x = \pi$

Hence, $\cos(x)$ is not normalized. It has norm $\|\cos(x)\| = \sqrt{\pi}$.

Similarly. The same calculation shows that $\cos(kx)$ and $\sin(kx)$ have norm $\sqrt{\pi}$ as well.

Example 187. How do we find, say, b_2?

Solution. Since the functions $1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$, the term $b_2\sin(2x)$ is the orthogonal projection of $f(x)$ onto $\sin(2x)$.

In particular, $b_2 = \frac{\langle f(x), \sin(2x)\rangle}{\langle \sin(2x), \sin(2x)\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}x$.

In conclusion:

A (nice) $f(x)$ on $[0, 2\pi]$ has the Fourier series

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

where

$$\begin{aligned} a_k = \frac{\langle f(x), \cos(kx)\rangle}{\langle \cos(kx), \cos(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t, \\ b_k = \frac{\langle f(x), \sin(kx)\rangle}{\langle \sin(kx), \sin(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)\mathrm{d}t, \\ a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t. \end{aligned}$$

How little we actually know!

Q: *How fast can we solve N linear equations in N unknowns?*

Estimated cost of Gaussian elimination:

$$\begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

- to create the zeros below the first pivot:
  $\Longrightarrow$ on the order of N^2 operations
- if there are N pivots total:
  $\Longrightarrow$ on the order of $N \cdot N^2 = N^3$ operations

- A more careful count places the cost at $\sim\frac{1}{3}N^3$ operations.
- For large N, it is only the N^3 that matters.

  It says that if $N \to 10N$ then we have to work 1000 times as hard.

That's not optimal! We can do better than Gaussian elimination:

- Strassen algorithm (1969): $N^{\log_2 7} = N^{2.807}$
- Coppersmith–Winograd algorithm (1990): $N^{2.375}$
- ... Stothers–Williams–Le Gall (2014): $N^{2.373}$ (If $N \to 10N$ then we have to work 229 times as hard.)

Is $N^{2+\text{(a tiny bit)}}$ possible? **We don't know!** (People increasingly suspect so.) (Better than N^2 is impossible; why?)

Comment. The above algorithms actually are for computing matrix products. It can be shown that, if $M(N)$ is the cost for multiplying two $N \times N$ matrices, then $N \times N$ systems can also be solved for cost on the order of $M(N)$. In other words, we don't even know how costly it is to multiply two matrices.

Good news for applications:

- Matrices typically have lots of structure and zeros

  which makes solving so much faster.

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Just for fun and curiousity!

Recall that we introduced the **dimension** of a vector space as the number of vectors in a/any basis. In Calculus, on the other hand, you learn about curves (1-dimensional), surfaces (2-dimensional) and solids (3-dimensional).

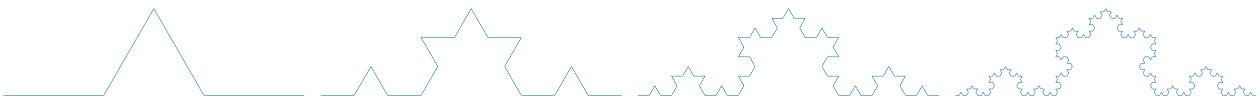
The reason that Linear Algebra is relevant for curved objects like surfaces is that locally these (typically) do look flat (like a plane), so that our tools apply at least locally.

What should a 1.5 dimensional thing look like?

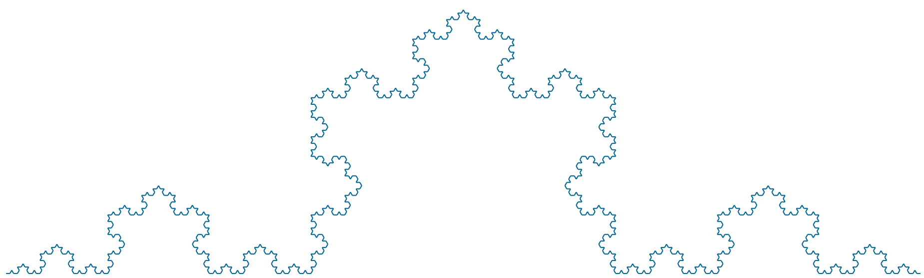
Something between a curve and a surface...

(Note that our linear algebra approach to dimension is not helpful.)

Here is a candidate.



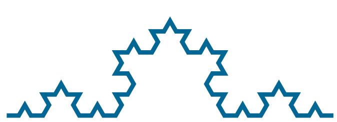
Continuing this process, results in the **Koch snowflake**, a **fractal**:



- Its perimeter is infinite!

  Why? At each iteration, the perimeter gets multiplied by $4/3$.

- The table below indicates that its boundary has dimension $\log_3(4) \approx 1.262$!!

	the effect of zooming in by a factor of 3		
		$\times 3$	$d = 1 = \log_3(3)$
		$\times 9$	$d = 2 = \log_3(9)$
		$\times 4$	$d = \log_3(4) \approx 1.262$

Does this have any practical relevance? Surprisingly, yes!

Have you ever wondered why perimeters of countries are missing from wikipedia? Or, why the coastline of the UK is listed as 11,000 miles by the UK mapping authority but 7,700 miles by the CIA Factbook?

Some of the fun can be found at: https://en.wikipedia.org/wiki/Coastline_paradox

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