(reflections) Suppose that M is the matrix for reflecting through the plane W in 3-space.

- The 1-eigenspace of M is W. (dimension 2)
- The -1-eigenspace of M is $W^\perp$. (dimension 1)

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get reflected to themselves. But those are precisely the vectors in the plane W (only vectors on the plane are unchanged by the reflection). On the other hand, the -1-eigenspace consists of those vectors $\boldsymbol{v}$ that get reflected to $-\boldsymbol{v}$ (the exact opposite direction). These are precisely the vectors orthogonal to the plane.

As in the case of projection matrices, because the eigenvalues are real and the eigenspaces are orthogonal, the reflection matrices are symmetric.

Comment. In this context, the line $W^\perp$ is often called the **normal line** of the plane W.

Example 112. Let A be the matrix for reflecting through the plane $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, -1$. The 1-eigenspace of A is W, and the -1-eigenspace is $W^\perp$.

In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

As in the previous example, $W^\perp = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is invertible (because 0 is not an eigenvalue).

Like any reflection matrix, A is symmetric.

Finally, note that $A^2 = I$ (reflecting twice isn't doing anything), so that $A^{-1} = A$. It follows that A is orthogonal, because $A^{-1} = A = A^T$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

An alternative way of computing reflection matrices. Realize that, if $\boldsymbol{n}$ is the vector orthogonal to the plane (i.e. $\boldsymbol{n}$ is the normal vector of the plane), then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{\boldsymbol{n}\cdot\boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ from I.

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Application: Linear differential equations

Example 113. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 114. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 115. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

> **(exponential function)** e^t is the unique solution to $y' = y$, $y(0) = 1$.
> From here, it follows that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$.

The latter is the Taylor series for e^t at $t = 0$ that we have seen in Calculus II.

Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^3}{3!} = \frac{t^2}{2!}$.

Example 116. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

Example 117. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.
Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at}y_0$.

Comment. It looks silly to write $e^{at}y_0$ instead of y_0e^{at} here, but we will soon replace the number a with a matrix A, and in that case only $e^{At}y_0$ makes sense.

Example 118. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

> **Theorem 119.** The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Recall from Example 117 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at}y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. Next time, we will define e^A by the familiar Taylor series for e^x.

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Definition 120. Let A be $n\times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Why? As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{At} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right] \\ &= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}.\end{aligned}$$

Therefore, $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$ indeed solves the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization!

Theorem 121. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^DP^{-1}$.

Why? Recall that, if $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.

$$\begin{aligned}e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)P^{-1} = Pe^DP^{-1}\end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for every "nice" function f. Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n\geqslant 0} a_n x^n$.

More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 122. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

Example 123. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $At = \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix}$, $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 124. (homework) Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Solution. (final solution only) $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$.

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Example 125. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, we diagonalize:

  For $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$. (That's homework!)

- We can then compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix} \end{aligned}$$

Comment. It is not necessary to compute $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$ (of course, you could do it, but that's more work).
Instead, recall that $A^{-1}\boldsymbol{b}$ is the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$. Here, solving $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we find $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Check. $\boldsymbol{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

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Example 126. We only discuss linear differential equations (DEs). Non-linear DEs include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$.

The order of a DE indicates the highest occuring derivative.

Note, however, that $y'' = \sin(t)y' + y$ is a linear DE, because y and its derivatives occur linearly.

We will see here how to solve those linear DEs which have constant coefficients. That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Review.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

- If we have the diagonalization $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$ (and $e^{At} = Pe^{Dt}P^{-1}$).

- If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 127. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(1-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

  Hence, the eigenvalues of A are $-1, 2$.

  The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} 2e^{-t} & -e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2e^{-t} + e^{2t} \\ e^{-t} - e^{2t} \end{bmatrix} \end{aligned}$$

Example 128. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Note. The "trick" of looking at the pair $\begin{bmatrix} y \\ y' \end{bmatrix}$ instead of a single function is what we used to translate the Fibonacci recurrence into a 2×2 system.

Example 129. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

The Jordan normal form

Note that we currently only know how to compute e^{At} when A is diagonalizable. Our next goal is to be able to compute the matrix exponential for all matrices.

Example 130. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$.

Solution. The eigenvalues of A are $4, 4$.

However, the 4-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 131. A **λ-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).
As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 132. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.

Example 133. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues 4, $4, 4$?

Solution. $\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$

The dimension of the 4-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you know from diagonalization).

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