Application: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.
They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing? Have a look at ratios of Fibonacci numbers:
$\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, $\frac{34}{21} \approx 1.619$, ...

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 100 below.

The crucial insight is the following simple observation:

$$F_{n+2} = F_{n+1} + F_n \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Everything we observe here about Fibonacci numbers holds for other sequences that satisfy similar recursion equations.

Theorem 100. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof.

- We already observed that thee recurrence $F_{n+2} = F_{n+1} + F_n$ translates into $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $T = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

  Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = T^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} &= PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix} \end{aligned}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula.

□

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Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1} \right)^n} \overset{n \to \infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 101. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution. $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$

Example 102. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -1$, $a_1 = 5$.

(a) Determine the first few terms of the sequence.

(b) Write down a matrix-vector version of the recursion.

(c) Find a Binet-like formula for a_n.

(d) Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $-1, 5, 7, 29, 79, 245, 727, 2189, 6559, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

(c) **(solution using matrix powers)** Thus, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

After some work (do it!), we diagonalize $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$ with $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}}_{= \begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3^{n+1} - 2(-1)^{n+1} \\ 3^n - 2(-1)^n \end{bmatrix}$.

In particular, $a_n = 3^n - 2(-1)^n$.

(simplified solution) The eigenvalues of $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ are 3 and -1.

Looking back at our work above, we can see that a_n therefore must have a formula of the form $a_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n$ for some unknown constants C_1, C_2 which we still need to figure out

Using the two initial conditions, we get two equations:

(a_0=) $C_1 + C_2 = -1$, ($a_1 =$) $3C_1 - C_2 = 5$.

Solving, we find $C_1 = 1$ and $C_2 = -2$ so that, in conclusion, $a_n = 3^n - 2(-1)^n$.

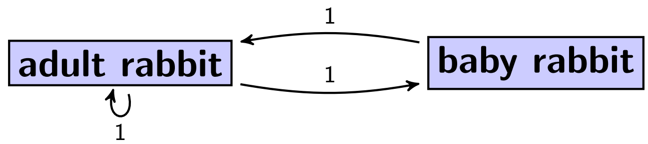
(d) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_1 = 0$.

Review. Fibonacci numbers, Binet formula

Example 103. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let a_t be the number of adult rabbit pairs after t months. Likewise, b_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} a_t + b_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

Definition 104. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called ***C*-finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $r_1 = r_2 = 1$.

In matrix-vector form.
$$\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{T} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$$

By the same reasoning as for Fibonacci numbers, C-finite sequences have a Binet-like formula:

Theorem 105. (generalized Binet formula) Suppose the recursion matrix T has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n$$

for certain numbers $C_1, \ldots, C_d$.

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$.

Comment. A little more care is needed in the case that eigenvalues are repeated.

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Corollary 106. Under the assumptions of the previous theorem, if λ_1 is the eigenvalue with the largest absolute value and $\lambda_1 > 0$, as well as $\alpha_1 \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1$.

Proof. This follows from $a_n = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n$ because, for large n, the term $C_1\lambda_1$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1} + \ldots + C_d\lambda_d^{n+1}}{C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n} = \frac{C_1\lambda_1 + C_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\lambda_d\left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\left(\frac{\lambda_d}{\lambda_1}\right)^n} \xrightarrow{n\to\infty} \frac{C_1\lambda_1}{C_1} = \lambda_1.$$

□

Example 107. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, \ldots$
Note that this sequence is C-finite of order 3.

(b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

Expanding by the 2nd row: $\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$

The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$
[You will not be asked to find roots of cubic polynomials by hand.]
Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.
Using the three initial conditions, we get three equations:
(a_0=) $C_1 + C_2 + C_3 = 0$, ($a_1 =$) $-C_1 + 2C_2 + 3C_3 = -2$, ($a_2 =$) $C_1 + 4C_2 + 9C_3 = 2$.
Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 108. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Example 109. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = C_1(1+\sqrt{6})^n + C_2(1-\sqrt{6})^n$ and we only need to figure out the values of C_1 and C_2.
Using the two initial conditions, we get two equations:
$(a_0 =)\ C_1 + C_2 = 0$, $(a_1 =)\ C_1(1+\sqrt{6}) + C_2(1-\sqrt{6}) = 1$.
Solving, we find $C_1 = \frac{1}{2\sqrt{6}}$ and $C_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did previously in the case of the Fibonacci numbers: starting with the recursion matrix T, we compute its diagonalization $T = PDP^{-1}$. Multiplying out $PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't need the Binet-like formula for this conclusion. Just the eigenvalues and the observation that C_1 cannot be 0 are enough. [We cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{6})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Another brief look at projections (and reflections)

(projections) Suppose that M is the projection matrix for projecting onto a subspace W.

- The 1-eigenspace of M is W.
- The 0-eigenspace of M is $W^\perp$.

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get projected to themselves. But those are precisely the vectors in W (recall that projecting a vector $\boldsymbol{v}$ onto W means producing the vector in W that is closest to $\boldsymbol{v}$). Can you likewise spell out the situation for the 0-eigenspace?

Note that M cannot have further eigenvalues (because the dimensions of W and $W^\perp$ already add up to the dimension of the space that we are working in).

Because the eigenvalues of M are real and the eigenspaces are orthogonal, the matrix M has a diagonalization of the form $M = PDP^T$ (make sure you can explain why!) which implies that M is symmetric (that's because $M^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = M$).

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Example 110. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^{-1}$.

(b) Diagonalize A as $A = PDP^T$.

Comment. This gives us yet another way to compute projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).

We already have a basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 0 & -1/4 \\ 0 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (of the 1-eigenspace), we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{17} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{17} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$.

Next, we normalize the vectors $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{17} \begin{bmatrix} -4 \\ 34 \\ 16 \end{bmatrix}, \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{357}} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}, \frac{1}{\sqrt{21}} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4/\sqrt{17} & -2/\sqrt{357} & -1/\sqrt{21} \\ 0 & 17/\sqrt{357} & -2/\sqrt{21} \\ 1/\sqrt{17} & 8/\sqrt{357} & 4/\sqrt{21} \end{bmatrix}$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{21} \begin{bmatrix} 20 & -2 & 4 \\ -2 & 17 & 8 \\ 4 & 8 & 5 \end{bmatrix}$ as in Example 54.

Example 111. Let A be the matrix for orthogonally projecting onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional). Note that we are lucky and already have an orthogonal basis for W. On the other hand, $W^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal. Like any projection matrix, A is symmetric.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.