**Example 70.** Using Gram–Schmidt, find an orthogonal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

**Solution.** We begin with the (not orthogonal) basis $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

We then construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$:

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \begin{pmatrix} \text{projection of} \\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1 \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\boldsymbol{q}_3 = \boldsymbol{w}_3 - \begin{pmatrix} \text{projection of } \boldsymbol{w}_3 \\ \text{onto span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\} \end{pmatrix} = \boldsymbol{w}_3 - \begin{pmatrix} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1 \end{pmatrix} - \begin{pmatrix} \text{projection of} \\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2 \end{pmatrix}$
  $= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

  Make sure you understand how $\boldsymbol{q}_3$ was designed to be orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$!

  Also note that breaking up the projection onto $\operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ into the projections onto $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ is only possible because $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthogonal.

Hence, $\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is an orthogonal basis of $W$.

**Important.** Normalizing, we obtain an orthonormal basis: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

**Example 71.** Determine the QR decomposition of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

**Solution.** The first step is Gram–Schmidt orthonormalization on the columns of $A$. We then use the resulting orthonormal vectors as the columns of $Q$.

We already did Gram–Schmidt in Example 70: from that work, we have $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

**Comment.** As commented earlier, the entries of $R$ have actually all been computed during Gram–Schmidt, so that, if we pay attention, we could immediately write down $R$ (no extra work required). Looking back at Example 70, can you see this?


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**Letting Sage do the work for us.**

```
Sage] A = matrix(QQbar, [[0,2,1],[3,1,1],[0,0,1],[0,0,1]])
```

```
Sage] A.QR(full=false)
```

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0.7071067811865475\,? \\ 0 & 0 & 0.7071067811865475\,? \end{bmatrix}, \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1.414213562373095\,? \end{bmatrix}\right)$$

**Comment.** Can you figure out what happens if you omit the `full=false`? Check out the comment under **Variations** for the QR decomposition in the previous lecture sketch. On the other hand, the `QQbar` is telling Sage to compute with algebraic numbers (instead of just rational numbers); if omitted, it would complain that square roots are not available

**Example 72. (extra)** Determine the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix}$.

**Solution.** We first apply Gram–Schmidt orthonormalization to the columns of $A$. For a variation, like a computer, we normalize after each step (rather than normalize at the end):

- $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \boldsymbol{q}_1\right)\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, so that $\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- $\boldsymbol{b}_3 = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_1\right)\boldsymbol{q}_1 - \left(\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_2\right)\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Therefore, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. Finally, $R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

In conclusion, we have found the QR decomposition: $\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}}_{R}$

**Comment.** As noted before, we actually could write down $R$ without any additional computation. Indeed, realize that the second column of $R$, that is $[2,3,0]^T$ means that

$$\text{2nd col of } A = 2\boldsymbol{q}_1 + 3\boldsymbol{q}_2.$$

Which we already knew from our computation of $\boldsymbol{q}_2$! Also, by construction, we know that the second column of $A$ is a linear combination of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ only, and that $\boldsymbol{q}_3$ enters the story later on. This corresponds to the fact that $R$ is always upper triangular.

**Letting Sage do the work for us.**

```
Sage] A = matrix(QQbar, [[1,2,4], [0,0,-5], [0,3,6]])
```

```
Sage] A.QR()
```

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

**Comment.** The `QQbar` is telling Sage to compute with algebraic numbers (instead of just rational numbers); in general, if omitted, it would complain that square roots are not available (because the matrices $Q$ and $R$ typically involve square roots). Here, we are lucky that square roots didn't creep in.

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**Example 73. (extra)** Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

**Solution. (final answer only)** $A = QR$ with $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ and $R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$.

**Example 74.** One practical application of the QR decomposition is solving systems of linear equations.

$$\begin{aligned} A\boldsymbol{x} = \boldsymbol{b} \quad &\Longleftrightarrow \quad QR\boldsymbol{x} = \boldsymbol{b} \qquad \text{(now, multiply with } Q^T \text{ from the left)} \\ &\Longrightarrow \quad R\boldsymbol{x} = Q^T\boldsymbol{b} \end{aligned}$$

The last system is triangular and can be solved by back substitution.

A couple of comments are in order:

- If $A$ is $n \times n$ and invertible, then the "$\Longrightarrow$" is actually a "$\Longleftrightarrow$".
- The equation $R\boldsymbol{x} = Q^T\boldsymbol{b}$ is always consistent! (Recall that $R$ is invertible.)

  Indeed, if $A$ is not $n \times n$ or not invertible, then $R\boldsymbol{x} = Q^T\boldsymbol{b}$ gives the least squares solutions!

  **Why?** $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b} \quad \Longleftrightarrow \quad \underbrace{(QR)^TQR}_{=R^TQ^TQR}\hat{\boldsymbol{x}} = (QR)^T\boldsymbol{b} \quad \Longleftrightarrow \quad R^TR\hat{\boldsymbol{x}} = R^TQ^T\boldsymbol{b} \quad \Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$

  [For the last step we need that $R$ is invertible, which is always the case when $A$ is $m \times n$ of rank $n$.]

- So, how does the QR way of solving linear systems compare to our beloved Gaussian elimination (LU)?

  It turns out that QR is a little slower than LU but makes up for it in "numerical stability".

  **What does that mean?** When computing numerically, we use floating point arithmetic and approximate each number by an expression of the form $0.1234 \cdot 10^{-16}$. A certain (fixed) number of bits is used to store the part $0.1234$ (here, $4$ decimal places of accuracy) as well as the exponent $-16$.

  Now, here is something terrible that can happen in numerical computations: mathematically, the quantities $x$ and $(x+1) - 1$ are exactly the same. However, numerically, they might not. Take, for instance, $x = 0.1234 \cdot 10^{-6}$. Then, to an accuracy of $4$ decimal places, $x + 1 = 0.1000 \cdot 10^1$, so that $(x+1) - 1 = 0.0000$. But $x \neq 0$. We completely lost all the information about $x$.

  To be numerically stable, an algorithm must avoid issues like that.

> $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
> $\Longleftrightarrow \quad R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$ (where $A = QR$)

**Review.** A matrix $A$ has orthonormal columns $\iff$ $A^T A = I$.

**Example 75.** Suppose $Q$ has orthonormal columns. What is the projection matrix $P$ for orthogonally projecting onto $\operatorname{col}(Q)$?

**Solution.** Recall that, to project onto $\operatorname{col}(A)$, the projection matrix is $P = A(A^T A)^{-1} A^T$.

Since $Q^T Q = I$, to project onto $\operatorname{col}(Q)$, the projection matrix is $P = Q Q^T$.

**Comment.** A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q}\cdot\boldsymbol{b})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

**Comment.** In particular, if $Q$ is not square, then $Q^T Q = I$ but $Q Q^T \neq I$. In some sense, $Q Q^T$ still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\operatorname{col}(Q)$ it does act like the identity for vectors in $\operatorname{col}(Q)$. (Vectors not in $\operatorname{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\operatorname{col}(Q)$.)

**Example 76.** Suppose $A$ is invertible. What is the projection matrix $P$ for orthogonally projecting onto $\operatorname{col}(A)$?

**Solution.** If $A$ is an invertible $n \times n$ matrix, then $\operatorname{col}(A) = \mathbb{R}^n$ (because the $n$ columns of $A$ are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\operatorname{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

**Definition 77.** An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to be orthonormal, not just orthogonal.]

| An $n \times n$ matrix $Q$ is orthogonal $\iff$ $Q^T Q = I$ |
|---|

In other words, $Q^{-1} = Q^T$.

**Example 78.** What can we say about $\det(Q)$ if $Q$ is orthogonal?

**Solution.** Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

## Review: Diagonalizability

**Example 79. (review)** If $A$ is a $2 \times 2$ matrix with $\det(A) = -8$ and eigenvalue $4$. What is the second eigenvalue?

**Solution.** Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is $-2$.

| $\det(A)$ is the product of the eigenvalues of $A$. |
|---|

**Why?** Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix $A$. We compute the characteristic polynomial $\det(A - \lambda I)$ and determine the $\lambda_i$ as the roots of that polynomial.

That means that we have the factorization $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_n - \lambda)\cdots(\lambda_n - \lambda)$. Now, set $\lambda = 0$ to conclude that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

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**Lemma 80.** A matrix $A$ is diagonalizable if and only if, for every eigenvalue $\lambda$ that is $k$ times repeated, the $\lambda$-eigenspace of $A$ has dimension $k$.

In short, an $n \times n$ matrix $A$ is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$ (i.e. "there are enough eigenvectors").

The next two examples illustrate that not all matrices are diagonalizable and that, even if a real matrix is diagonalizable, the eigenvalues and eigenvectors might be complex.

**Example 81.** What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Is $A$ diagonalizable?

**Solution.** The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2$, which has $\lambda = 0$ as a double root.

However, the $0$-eigenspace $\text{null}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is only $1$-dimensional.

As a consequence, $A$ is not diagonalizable.

**Example 82.** What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$? Is $A$ diagonalizable?

**Solution.** The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.

Hence, the eigenvalues are $\pm i$.

The $i$-eigenspace $\text{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

The $-i$-eigenspace $\text{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, $A$ has the diagonalization $A = PDP^{-1}$ with $D = \begin{bmatrix} i & \\ & -i \end{bmatrix}$ and $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$.

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**Example 83. (review)** In Example 17, we diagonalized $A=\begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ as $A=PDP^{-1}$.

We found that one choice for $P$ and $D$ is $P=\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Spell out what that tells us about $A$!

**Solution.** The diagonal entries $5,2,2$ of $D$ are the eigenvalues of $A$.
The columns of $P$ are corresponding eigenvectors of $A$.

- $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a 5-eigenvector of $A$ (that is, $A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}=5\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$).
- The 2-eigenspace of $A$ is 2-dimensional. A basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

## The spectral theorem

Recall that a matrix $A$ is symmetric if and only if $A^T=A$.

**Theorem 84. (spectral theorem, long version)** Suppose $A$ is a symmetric matrix.

- $A$ is always diagonalizable.
- All eigenvalues of $A$ are real.
- The eigenspaces of $A$ are orthogonal.

**Proof.** We will prove (parts of) the spectral theorem later on. For now, we just appreciate that the spectral theorem guarantees all these nice things to happen for symmetric matrices (for any specific $A$ we know how to determine whether $A$ is diagonalizable and what its eigenspaces are).

**Comment.** The eigenspaces of $A$ being orthogonal means that eigenvectors for different eigenvalues are always orthogonal.

**Important consequence.** In the diagonalization $A=PDP^{-1}$, we can choose $P$ to be orthogonal (in which case $P^{-1}=P^T$). In that case, the diagonalization takes the special form $A=PDP^T$, where $P$ is orthogonal and $D$ is diagonal.

**(spectral theorem, compact version)** A symmetric matrix $A$ can always be diagonalized as $A=PDP^T$, where $P$ is orthogonal and $D$ is diagonal (and both are real).

**How?** We proceed as in the diagonalization $A=PDP^{-1}$. For a symmetric matrix $A$, we can arrange $P$ to be orthogonal, by normalizing its columns. If there is a repeated eigenvalue, then we also need to make sure to pick an orthonormal basis for the corresponding eigenspace (for instance, using Gram–Schmidt).

**Advanced comment.** A matrix such that $A^TA=AA^T$ is called **normal**. For normal matrices, the (complex!) eigenspaces are again orthogonal to each other. However, normal matrices which are not symmetric will always have complex eigenvalues. (In that case, the orthogonal matrix $P$ gets replaced with a unitary matrix, the complex version of orthogonal matrices, and the $P^T$ becomes the conjugate transpose $P^*=\bar{P}^T$.)

**Example 85.**

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(b) Diagonalize $A$ as $A = PDP^T$.

**Solution.**

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so $A$ has eigenvalues $4, -2$.

The $4$-eigenspace is $\text{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The $-2$-eigenspace is $\text{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

**Important observation.** The $4$-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the $-2$-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthogonal!

**Review.** The product of all eigenvalues $-2 \cdot 4 = -8$ equals the determinant $\det(A) = 1 - 9 = -8$.

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose $P$ so that $P^{-1} = P^T$, which means that $P$ must be **orthogonal** (meaning orthonormal columns). [Choosing such a $P$ is only possible if the eigenspaces of $A$ are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, we then have $A = PDP^T$.

**Example 86. (again, simplified)** Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ as $A = PDP^T$.

**Solution.** See Example 85 for a solution that illustrates how to diagonalize any symmetric matrix. For a simplified solution, note that we can see right away that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a $4$-eigenvector (since the row sums are equal!).

Because the eigenspaces are orthogonal (since $A$ is symmetric!), $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ must also be an eigenvector.

Indeed, $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ shows that the corresponding eigenvalues is $-2$.

We normalize the two eigenvectors and use them as the columns of $P$, so that $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is an orthogonal matrix ($P^{-1} = P^T$). With $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ we then have $A = PDP^T$.

**Example 87.** Let $A$ be a symmetric $2 \times 2$ matrix with $7$-eigenvector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\det(A) = -21$. Determine the second eigenvalue and a corresponding eigenvector.

**Solution.** $A$ has $-\frac{21}{7} = -3$-eigenvector $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

**Comment.** Recall that, because $A$ is symmetric, the eigenvector must be orthogonal to $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

[In general, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} -b \\ a \end{bmatrix}$ are orthogonal.]

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