pseudoinverse of $\underset{m\times n}{A}$

$= $ the matrix A^+ ($n\times m$) so that $Ax=b$ has "optimal" solution $x=A^+b$

smallest least-squares

- if A invertible, $A^+ = A^{-1}$
- if A has full column rank, $A^+ = (A^TA)^{-1}A^T$.
- if $A = U\Sigma V^T$ (SVD), then
$$A^+ = V\Sigma^+U^T$$

$(AB)^{-1} = B^{-1}A^{-1}$

Why? $Ax=b \iff \Sigma y = U^Tb$

$U\Sigma \underbrace{V^Tx}_{=:y} = b$ optimal sol: $y=\Sigma^+U^Tb$

$x=Vy$

x,y same norm $x=Vy=\underbrace{\boxed{V\Sigma^+U^T}}_{A^+}b$

EG $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Option 1: A full column rank

$$\Rightarrow A^+ = (A^TA)^{-1}A^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Option 2: $A = U\Sigma V^T$

with $U = \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow A^+ = V\Sigma^+U^T$$

with $\Sigma^+ = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$