pseudo inverse of $\underset{m\times n}{A}$

$:=$ the matrix A^+ so that

$Ax = b$ has "optimal" solution $x = A^+ b$

smallest least-squares

EG • if A is invertible, then $A^+ = A^{-1}$

• if A has full column rank then $A^+ = (A^TA)^{-1}A^T$

review:

$Ax = b$

to find least-squares solutions:

$A^TAx = A^Tb$

$\rightsquigarrow x = \boxed{(A^TA)^{-1}A^T}\, b$

A^+

EG $\underset{3\times 2}{\Sigma} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$ full column rank

2×2

$\Rightarrow \Sigma^+ = (\Sigma^T\Sigma)^{-1}\Sigma^T$

$= \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$

$\begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix}$

EG $\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$ not full column rank

$\Sigma x = b \qquad \Sigma \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \begin{matrix} 2x_1 = b_1 \\ 3x_2 = b_2 \end{matrix}$

$\rightsquigarrow$ general sol: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1/2 \\ b_2/3 \\ t \end{bmatrix}$

free

"optimal" sol: $\begin{bmatrix} b_1/2 \\ b_2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Σ^+

Σ diagonal $m \times n$

$\Rightarrow \Sigma^+$ diagonal $n \times m$ with non-zero entries inverted