Solving recurrence equations

EG $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ $\quad a_0 = 0 \quad a_1 = -2 \quad a_2 = 2$

recurrence of order 3 $\qquad$ initial conditions

first few terms

$$a_3 = 4a_2 - a_1 - 6a_0 = 10$$
$$a_4 = 4a_3 - a_2 - 6a_1 = 50 \quad \dots$$

matrix-vector version of recursion

$$\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4a_{n+2} - a_{n+1} - 6a_n \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{M} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$$

$$\begin{matrix} + & - & + & \dots \\ - & + & & \\ \vdots & & & \end{matrix}$$

$$\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \underbrace{\begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix}}_{\lambda+6} - \lambda \cdot \underbrace{\begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix}}_{-\lambda(4-\lambda)} = -\lambda^3 + 4\lambda^2 - \lambda - 6$$

$\Rightarrow M$ has eigenvalues $-1, 2, 3$

Binet-like formula $\quad a_n = c_1 (-1)^n + c_2\, 2^n + c_3\, 3^n$

$$\begin{aligned} a_0 &= 0 : & c_1 + c_2 + c_3 &= 0 \\ a_1 &= -2 : & -c_1 + 2c_2 + 3c_3 &= -2 \\ a_2 &= 2 : & c_1 + 4c_2 + 9c_3 &= 2 \end{aligned} \quad \xrightarrow[\text{work!}]{} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -2 \\ c_3 &= 1 \end{aligned}$$

$$a_n = (-1)^n - 2 \cdot 2^n + 3^n$$

if $c_3 \neq 0$

check $\quad a_3 = (-1)^3 - 2 \cdot 2^3 + 3^3 = 10 \quad ✓$

dominates $2^n, (-1)^n$

for large n:

$$a_n \approx c_3\, 3^n$$

$$\frac{a_{n+1}}{a_n} \approx \frac{c_3\, 3^{n+1}}{c_3\, 3^n} = 3$$

growth $\quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$