Recurrence equations

Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ($a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$)

characterized by $a_{n+2} = a_{n+1} + a_n$ (recurrence) and $a_0 = 0,\ a_1 = 1$ (initial conditions)

growth? ratios: $\frac{a_2}{a_1} = 1 \quad \frac{a_3}{a_2} = 2 \quad \frac{a_4}{a_3} = 1.5 \quad \frac{a_5}{a_4} \approx 1.67 \quad \frac{a_6}{a_5} = 1.6$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{golden ratio}$$

Binet formula

$$a_n = \frac{1}{\sqrt{5}}\left[\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^n}_{1.618^n} - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{(-0.618)^n}\right]$$

EG $a_{n+2} = 2a_{n+1} + 3a_n \qquad a_0 = -1,\ a_1 = 5$

first few terms $a_2 = 2a_1 + 3a_0 = 7, \quad a_3 = 2a_2 + 3a_1 = 29, \ \ldots$

matrix-vector version of recursion

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2a_{n+1} + 3a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

solving recurrence using matrix powers

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$$

$$D = \begin{bmatrix} 3 & \\ & -1 \end{bmatrix} \quad P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= PD^nP^{-1} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & \\ & (-1)^n \end{bmatrix}}_{= \begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$$= \begin{bmatrix} 3^{n+1} - 2\cdot(-1)^{n+1} \\ 3^n - 2\cdot(-1)^n \end{bmatrix}$$

simplified solution

$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $3, -1$

$$\Longrightarrow \quad a_n = c_1 \cdot 3^n + c_2 \cdot (-1)^n$$

find c_1, c_2 using initial conditions:

$a_0 = -1: \quad c_1 + c_2 = -1 \quad \rightsquigarrow \quad c_1 = 1$

$a_1 = 5: \quad 3c_1 - c_2 = 5 \qquad\quad c_2 = -2$

$$a_n = 3^n - 2\cdot(-1)^n$$

Binet-like formula