Markov chains

(discrete dynamical systems)

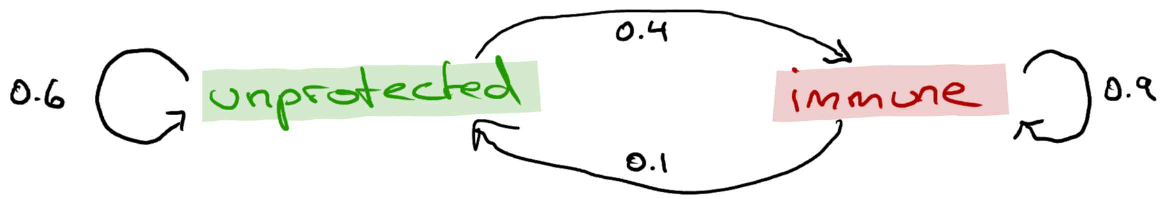
EG Fixed population of people with or without active immunization against, say, tetanus.

Each year:

40% of those unprotected get vaccinated

10% of those with immunization lose protection

What is the immunization rate in the long term?



x_t : proportion of population unprotected at time t (in years)

y_t : —— " —— immune at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6\,x_t + 0.1\,y_t \\ 0.4\,x_t + 0.9\,y_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}}_{\text{transition matrix } M} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

work!

eigenvalues of M are $1, \frac{1}{2}$

$$= \frac{1}{5} \begin{bmatrix} 1 + 4\cdot 0.5^n & 1 - 0.5^n \\ 4 - 4\cdot 0.5^n & 4 + 0.5^n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$0.5^n \to 0$ as $n \to \infty$

$$\xrightarrow[\text{as } n\to\infty]{} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \overbrace{x_0 + y_0}^{=1} \\ 4x_0 + 4y_0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{equilibrium state } \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$$

$\Rightarrow$ long term immunization rate $= \frac{4}{5} = 80\%$

easier way to find equilibrium state $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

if there is an equilibrium

$$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = M \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$$

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ 1-eigenvector of M

1-eigenspace of M = null $\begin{pmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{pmatrix}$ basis: $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

since $x_\infty + y_\infty = 1$, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$