Spectral theorem

EG $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = PDP^{-1}$

A symmetric: $A^T = A$

with $D = \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix}$ $P = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

- eigenvalues $\underbrace{7, 7}_{\text{multiplicity } 2}, -2$
- 7-eigenspace $= \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- -2-eigenspace $= \text{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$

important observation: 7-eigenspace $\perp$ -2-eigenspace

SPECTRAL THEOREM If A is $n \times n$ and symmetric, then:

- A is diagonalizable,
- the eigenvalues of A are real,
- the eigenspaces of A are orthogonal to each other.

compact version A is $n \times n$ and symmetric

$\iff A = PDP^T$ for some D diagonal (real!) and P orthogonal

$\iff P^{-1} = P^T$

Why? choose orthonormal eigenvectors to construct P

EG $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ symmetric eigenvalues: $4, -1$

- 4-eigenspace $= \text{null}\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$ basis: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ normalized: $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- -1-eigenspace $= \text{null}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ basis: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

orthogonal!

$A = PD\underbrace{P^T}_{= P^{-1}}$ with $D = \begin{bmatrix} 4 & \\ & -1 \end{bmatrix}$ $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ orthogonal

EG $\quad A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

- eigenvalues $\underbrace{7, 7}_{\text{multiplicity } 2}, -2$
- 7-eigenspace $= \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- -2-eigenspace $= \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$

goal: Diagonalize A as $A = PDP^T$.

need: orthonormal basis of eigenvectors

- 7-eigenspace:

  Gram-Schmidt on $w_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

  $q_1 = w_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

  $q_2 = w_2 - \dfrac{w_2 \cdot q_1}{q_1 \cdot q_1} q_1 = \frac{5}{5}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

  orthonormal basis: $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{45}}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

- -2-eigenspace:

  orthonormal basis: $\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

$\Longrightarrow \quad A = PDP^T$ with $D = \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix}$

$$P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$$

orthogonal