Diagonalization

THM Suppose that A is $n \times n$ with n independent eigenvectors $x_1, x_2, \dots, x_n$.
Then A is **diagonalizable**:

$$A = P D P^{-1}$$

P: cols are eigenvectors
D: diagonal matrix with eigenvalues on diagonal

Why?

$$A x_i = \lambda_i x_i \qquad A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$A P_{P^{-1}} = P D_{P^{-1}} \qquad = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

EG Diagonalize $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

- characteristic polynomial
  $$\det(A - \lambda I) \underset{\text{work}}{=} (2-\lambda)^2 (5-\lambda)$$
- eigenvalues: $5, \underbrace{2, 2}_{\text{multiplicity } 2}$
- 5-eigenspace has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$
- 2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\implies A = P D P^{-1} \qquad P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$