Homework Set 12

Problem 1

Example 22. Find the best approximation of $f(x) = x^2$ on the interval $[1, 4]$ using a function of the form $y = a\sqrt{x}$.

Solution. Because we are working with functions on $[1, 4]$, the dot product between two functions is

$$\langle f(x), g(x)\rangle = \int_1^4 f(t)g(t)\mathrm{d}t.$$

The best approximation is the orthogonal projection of x^2 onto $\operatorname{span}\{\sqrt{x}\}$, which is

$$\begin{aligned}\frac{\langle x^2, \sqrt{x}\rangle}{\langle \sqrt{x}, \sqrt{x}\rangle}\sqrt{x} &= \frac{\int_1^4 t^2 \cdot \sqrt{t}\mathrm{d}t}{\int_1^4 \sqrt{t}\cdot\sqrt{t}\mathrm{d}t} \cdot \sqrt{x} = \frac{\int_1^4 t^{5/2}\,\mathrm{d}t}{\int_1^4 t\mathrm{d}t} \cdot \sqrt{x} \\ &= \frac{\left[\frac{1}{7/2}t^{7/2}\right]_1^4}{\left[\frac{1}{2}t^2\right]_1^4} \cdot \sqrt{x} = \frac{\frac{2}{7}\cdot 4^{7/2} - \frac{2}{7}}{\frac{1}{2}\cdot 4^2 - \frac{1}{2}} \cdot \sqrt{x} = \frac{\frac{254}{7}}{\frac{15}{2}} \cdot \sqrt{x} = \frac{508}{105}\sqrt{x}.\end{aligned}$$

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Problem 2

Example 23. Find the best approximation of $f(x) = x^4$ on the interval $[1, 3]$ using a function of the form $y = a + bx$.

Solution. First, note that this best approximation is the orthogonal projection of x^4 onto $\mathrm{span}\{1, x\}$. However, this orthogonal projection is not simply the projection onto 1 plus the projection onto x. That's because 1 and x are not orthogonal (as functions on $[1, 3]$):

$$\langle 1, x\rangle = \int_1^3 t\mathrm{d}t = \left[\frac{1}{2}t^2\right]_1^3 = \frac{3^2}{2} - \frac{1}{2} = 4 \neq 0.$$

To find an orthogonal basis for $\mathrm{span}\{1, x\}$, following Gram–Schmidt, we compute

$$x - \begin{pmatrix}\text{projection of} \\ x \text{ onto } 1\end{pmatrix} = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - \frac{\int_1^3 t\mathrm{d}t}{\int_1^3 1\mathrm{d}t} = x - \frac{4}{2} = x - 2.$$

Hence, $1, x - 2$ is an orthogonal basis for $\mathrm{span}\{1, x\}$.

The desired orthogonal projection of x^4 onto $\mathrm{span}\{1, x\} = \mathrm{span}\{1, x - 2\}$ therefore is

$$\frac{\langle x^4, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x^4, x - 2\rangle}{\langle x - 2, x - 2\rangle}(x - 2) \quad = \quad \frac{\int_1^3 t^4\mathrm{d}t}{\int_1^3 1\mathrm{d}t} + \frac{\int_1^3 t^4(t - 2)\mathrm{d}t}{\int_1^3 (t - 2)^2\mathrm{d}t}(x - 2)$$

We compute the three new integrals:

$$\begin{aligned}
\int_1^3 t^4\mathrm{d}t &= \left[\frac{1}{5}t^5\right]_1^3 = \frac{3^5}{5} - \frac{1}{5} = \frac{242}{5} \\
\int_1^3 (t - 2)^2\mathrm{d}t &= \int_1^3 (t^2 - 4t + 4)\mathrm{d}t = \left[\frac{1}{3}t^3 - 2t^2 + 4t\right]_1^3 = 3 - \frac{7}{3} = \frac{2}{3} \\
\int_1^3 t^4(t - 2)\mathrm{d}t &= \int_1^3 (t^5 - 2t^4)\mathrm{d}t = \left[\frac{1}{6}t^6 - \frac{2}{5}t^5\right]_1^3 = \frac{243}{10} - \left(-\frac{7}{30}\right) = \frac{368}{15}
\end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_1^3 t^4\mathrm{d}t}{\int_1^3 1\mathrm{d}t} + \frac{\int_1^3 t^4(t - 2)\mathrm{d}t}{\int_1^3 (t - 2)^2\mathrm{d}t}(x - 2) \quad = \quad \frac{\frac{242}{5}}{2} + \frac{\frac{368}{15}}{\frac{2}{3}}(x - 2) = \frac{121}{5} + \frac{184}{5}(x - 2) = \frac{184}{5}x - \frac{247}{5}.$$