Homework Set 11

Problem 1

Example 17. Determine the pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}$.

Solution. For such diagonal matrices, we only need to invert the diagonal entries and transpose the dimensions.
$A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/5 \\ 0 & 0 \end{bmatrix}$

Problem 2

Example 18. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$ (without computing the SVD first).

Solution. This matrix clearly has full column rank (because the two columns are not multiples of each other).
Hence, $A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 13 & -6 \\ -6 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 8 & 12 & 39 \\ -27 & 26 & 18 \end{bmatrix}$.

Problem 3

Example 19. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$ (by computing the SVD first).

Solution. See Example 166 in the lecture notes on how to avoid much of the below computations. Here, we instead embrace the opportunity to practice. We first compute the SVD of A:

First, we need to diagonalize $A^T A = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}$. Let us write $|A|$ for $\det(A)$:

$$\begin{aligned} \begin{vmatrix} 4-\lambda & -4 & 2 \\ -4 & 4-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} &= (4-\lambda) \cdot \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - (-4) \cdot \begin{vmatrix} -4 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} -4 & 4-\lambda \\ 2 & -2 \end{vmatrix} \\ &= (4-\lambda) \cdot (\lambda^2 - 5\lambda) + 4 \cdot (4\lambda) + 2 \cdot (2\lambda) = -\lambda^3 + 9\lambda^2 = \lambda^2 (9 - \lambda) \end{aligned}$$

Hence, the eigenvalues of $A^T A$ are $9, 0, 0$.

- $\lambda = 9$: $\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \underset{R_3 + \frac{2}{5} R_1 \Rightarrow R_3}{\overset{R_2 - \frac{4}{5} R_1 \Rightarrow R_2}{\rightsquigarrow}} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & -\frac{18}{5} & -\frac{36}{5} \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix}$
  $\underset{-\frac{5}{9} R_2 \Rightarrow R_2}{\overset{-\frac{1}{5} R_1 \Rightarrow R_1}{\rightsquigarrow}} \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - \frac{4}{5} R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

  Hence, the 9-eigenspace has basis $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

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- $\lambda = 0$: $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{\substack{R_2 + R_1 \Rightarrow R_2 \\ R_3 - \frac{1}{2}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  Hence, the 0-eigenspace has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$. For the SVD we have to turn this basis into an orthogonal one.

  Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

Thus $A^T A = P D P^T$ with $D = \begin{bmatrix} 9 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

[We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V \Sigma^2 V^T$, we conclude that $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = 1$.

Hence, $A = U \Sigma V^T$ with $U = \begin{bmatrix} 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$.

Using the SVD of A, we can easily obtain its pseudoinverse:

$$A^+ = V \Sigma^+ U^T = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Comments. This was good practice computing SVDs but we did a lot of work that we could have simplified: Can you see why it was clear that $A^T A$ was going to have 0 as a repeated eigenvalue? Can you see why the last two columns of P are irrelevant in our computation? Can you see how we could have obtained the first column of P without computation? [Also, can you argue geometrically why the pseudoinverse is what it is?]

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Problem 4

Example 20. Find the smallest norm solution to $4x_1 + 3x_2 + 5x_3 = 3$.

Solution. If $A = [\ 4\ \ 3\ \ 5\ ]$, then the smallest norm solution is $\boldsymbol{x} = A^+[\ 3\ ]$.

From earlier computations (see Example 167) we know that $A^+ = \frac{1}{4^2+3^2+5^2}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Hence, the smallest norm solution is $\boldsymbol{x} = A^+[\ 3\ ] = \frac{3}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Problem 5

Example 21. Determine the best rank 1 approximation of $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution. We first compute the SVD of A:

- First, we need to diagonalize $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\right) = (2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$

  Hence, the eigenvalues of A^TA are $6, 1$.

  - $\lambda = 6$: $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \overset{R_2 - \frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{4}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

    Hence, the 6-eigenspace has basis $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

  - $\lambda = 1$: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \overset{R_2 + 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

    Hence, the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 6 & \\ & 1 \end{bmatrix}$ and $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

  [We had to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2V^T$, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}$.

  For the rank 1 approximation, we only need the first column of U, so we stop here.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} -5/\sqrt{30} & * & * \\ -2/\sqrt{30} & * & * \\ -1/\sqrt{30} & * & * \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

From the SVD of A, we obtain the best rank 1 approximation by only using the first columns of U and V (and truncating Σ to a 1×1 matrix):

Thus, the best rank 1 approximation of A is $\frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}[\ \sqrt{6}\ ]\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}^T = \sqrt{\frac{6}{30 \cdot 5}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}[\ -1\ \ 2\ ] = \frac{1}{5}\begin{bmatrix} 5 & -10 \\ 2 & -4 \\ 1 & -2 \end{bmatrix}$.

Comment. Like for U, we could have omitted the computation of the 1-eigenvector (second column of V).