Problem 1

Example 7. How many different Jordan normal forms are there for a 6×6 matrix with eigenvalues $4, 4, 4, 4, 9, 9$?

  Solution. One eigenvalue has multiplicity 4, the other multiplicity 2.

  Therefore, there are $5 \cdot 2 = 10$ possible Jordan normal forms.

  [See Example 135 in Lecture 25 for where the 5 and 2 come from. Alternatively, you can see this from the listing that follows below.]

  Listing all of them.

$$\begin{bmatrix} 4 & & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & 1 & & \\ & & & 4 & & \\ & & & & 9 & \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & 1 & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & 1 & & & \\ & & 4 & 1 & & \\ & & & 4 & & \\ & & & & 9 & \\ & & & & & 9 \end{bmatrix},$$

$$\begin{bmatrix} 4 & & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & 1 \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & 1 \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & & & & \\ & & 4 & 1 & & \\ & & & 4 & & \\ & & & & 9 & 1 \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & 1 & & & \\ & & 4 & & & \\ & & & 4 & & \\ & & & & 9 & 1 \\ & & & & & 9 \end{bmatrix},\ \begin{bmatrix} 4 & 1 & & & & \\ & 4 & 1 & & & \\ & & 4 & 1 & & \\ & & & 4 & & \\ & & & & 9 & 1 \\ & & & & & 9 \end{bmatrix}$$

Problem 2

Example 8. How many different Jordan normal forms are there for a 14×14 matrix with eigenvalues $3, 4, 4, 4, 4, 5, 5, 5, 5, 7, 7, 7, 7, 8$?

  Solution. The multiplicities of the eigenvalues are $1, 4, 4, 4, 1$.

  Hence, there are $1 \cdot 5 \cdot 5 \cdot 5 \cdot 1 = 125$ possible Jordan normal forms.

Problem 3

Example 9. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

  Solution. We can see right away that $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is not diagonalizable (because it is a 2×2 Jordan block).

  The solution to the differential equation is

$$\begin{aligned} \boldsymbol{y}(t) &= e^{At} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= e^{5It + Nt} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= e^{5It} e^{Nt} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } 5It \text{ and } Nt \text{ commute)} \\ &= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \cdots \right) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} (1 + Nt) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } N^2 = \boldsymbol{0}) \\ &= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \begin{bmatrix} 1 - 2t \\ -2 \end{bmatrix} = \begin{bmatrix} (1 - 2t)e^{5t} \\ -2e^{5t} \end{bmatrix}. \end{aligned}$$

Problem 4

Example 10. Determine the 2×2 matrix Q for rotation by 49 degrees.

Solution. Recall that the rotation matrix for angle θ is $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (see Example 145 in Lecture 27).

Hence, $Q = \begin{bmatrix} \cos(49°) & -\sin(49°) \\ \sin(49°) & \cos(49°) \end{bmatrix} = \begin{bmatrix} \cos\left(49 \cdot \frac{2\pi}{360}\right) & -\sin\left(49 \cdot \frac{2\pi}{360}\right) \\ \sin\left(49 \cdot \frac{2\pi}{360}\right) & \cos\left(49 \cdot \frac{2\pi}{360}\right) \end{bmatrix} \approx \begin{bmatrix} 0.6561 & -0.7547 \\ 0.7547 & 0.6561 \end{bmatrix}$.

Problem 5

Example 11. Determine the square of the norm of the vector $\begin{bmatrix} 5 - 5i \\ 1 + 3i \end{bmatrix}$.

Solution. $\left\| \begin{bmatrix} 5 - 5i \\ 1 + 3i \end{bmatrix} \right\|^2 = |5 - 5i|^2 + |1 + 3i|^2 = (5^2 + (-5)^2) + (1^2 + 3^2) = 50 + 10 = 60$

Problem 6

Example 12. Given $A = \begin{bmatrix} 2 - i & -1 + i \\ 3 - 3i & 2 + 3i \end{bmatrix}$, determine A^*.

Solution. $A^* = (\bar{A})^T = \begin{bmatrix} 2 + i & -1 - i \\ 3 + 3i & 2 - 3i \end{bmatrix}^T = \begin{bmatrix} 2 + i & 3 + 3i \\ -1 - i & 2 - 3i \end{bmatrix}$

Problem 7

Example 13. Invert the complex number $5 - 4i$.

Solution. $\dfrac{1}{5 - 4i} = \dfrac{5 + 4i}{(5 - 4i)(5 + 4i)} = \dfrac{5 + 4i}{5^2 - (4i)^2} = \dfrac{5 + 4i}{25 + 16} = \dfrac{5}{41} + \dfrac{4}{41}i$