Sketch of Lecture 36

Wed, 4/29/2020

Linear transformations

Throughout, V and W are vector spaces.

Just like we went from column vectors to abstract vectors (such as polynomials), the concept of a matrix leads to abstract linear transformations.

In the other direction, picking a basis, abstract vectors can be represented as column vectors (see Lecture 35). Correspondingly, linear transformations can then be represented as matrices.

Definition 175. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It necessarily sends the zero vector in V to the zero vector in W:

- $T(\mathbf{0}) = \mathbf{0}$ [because $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$]

Comment. Linear transformations are special functions and, hence, can be composed. For instance, if $T: V \to W$ and $S: U \to V$ are linear transformations, then $T \circ S$ is a linear transformation $U \to W$ (sending $\boldsymbol{x}$ to $T(S(\boldsymbol{x}))$). If S, T are represented by matrices A, B, then $T \circ S$ is represented by the matrix BA. In other words, matrix multiplication arises as the composition of (linear) functions.

Example 176. The **derivative** you know from Calculus I is linear.

Indeed, the map $D: \left\{\begin{array}{c}\text{space of all} \\ \text{differentiable} \\ \text{functions}\end{array}\right\} \to \left\{\begin{array}{c}\text{space of all} \\ \text{functions}\end{array}\right\}$ defined by $f(x) \mapsto f'(x)$ is a linear transformation:

- $\underbrace{D(f(x) + g(x))}_{(f(x)+g(x))'} = \underbrace{D(f(x))}_{f'(x)} + \underbrace{D(g(x))}_{g'(x)}$
- $\underbrace{D(cf(x))}_{(cf(x))'} = \underbrace{cD(f(x))}_{cf'(x)}$

These are among the first properties you learned about the derivative.

Similarly, the **integral** you love from Calculus II is linear:

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x, \qquad \int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$$

In this form, we are looking at a map $T: \left\{\begin{array}{c}\text{space of all} \\ \text{continuous} \\ \text{functions}\end{array}\right\} \to \mathbb{R}$ defined by $T(f(x)) = \int_a^b f(x)\mathrm{d}x$.

Example 177. Consider the space V of all polynomials $p(x)$ of degree 3 or less. The map $D: V \to V$ given by $p(x) \mapsto p'(x)$ is a linear. Write down the matrix M for this linear map with respect to the basis $1, x, x^2, x^3$.

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Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = 2x$.

Example 178. Consider the map

$$D: \left\{ \begin{array}{c} \text{space of poly's} \\ \text{of degree} \leqslant 3 \end{array} \right\} \to \left\{ \begin{array}{c} \text{space of poly's} \\ \text{of degree} \leqslant 2 \end{array} \right\}, \quad p(x) \mapsto p'(x).$$

Write down the matrix M for this linear map with respect to the bases $1, x, x^2, x^3$ and $1, x, x^2$.

Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 = 2x$.

Example 179. What is the pseudo-inverse of the matrix M from the previous example. Interpret your finding.

Solution. (final answer only) The pseudo-inverse of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

The corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$ and x^2 to $\frac{1}{3}x^3$. That is, the pseudo-inverse computes the antiderivative of each monomial.

Comment. This is not surprising, since we are familiar from Calculus with the concepts of derivatives and antiderivatives (or integrals), and that these are "pseudo" inverse to each other.

Comment. Similarly, the pseudo-inverse of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$.

Now, the corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$, x^2 to $\frac{1}{3}x^3$, and x^3 to 0. That is, the pseudo-inverse computes the antiderivative of each monomial, with the exception of x^3 which gets send to 0 (its antiderivative does not live in the space of polynomials of degree 3).

Example 180. (The April Fools' Day "proof" that $\pi = 4$, cont'd)

In that "proof", we are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm introduced as we did for functions.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as you learned in Calculus II, the arc length of a function $y = f(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f'(x)^2} \mathrm{d}x.$$

Observe that this involves f'. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x \right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

Fourier series

A **Fourier series** for a function $f(x)$ is a series of the form

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

You may have seen Fourier series in other classes before. Our goal here is to tie them in with what we have learned about orthogonality.

In these other classes, you would have seen formulas for the coefficients a_k and b_k. We will see where those come from.

Observe that the right-hand side combination of cosines and sines is 2π-periodic.

Let us consider (nice) functions on $[0, 2\pi]$.

Or, equivalently, functions that are 2π-periodic.

We know that a natural inner product for that space of functions is

$$\langle f, g \rangle = \int_0^{2\pi} f(t) g(t) \mathrm{d}t.$$

Example 181. Show that $\cos(x)$ and $\sin(x)$ are orthogonal (in that sense).

Solution. $\langle \cos(x), \sin(x) \rangle = \int_0^{2\pi} \cos(t) \sin(t) \mathrm{d}t = \left[\frac{1}{2} (\sin(t))^2 \right]_0^{2\pi} = 0$

In fact:

$\boxed{\text{All the functions } 1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots \text{ are orthogonal to each other!}}$

Moreover, they form a basis in the sense that every other (nice) function can be written as a (infinite) linear combination of these basis functions.

Example 182. What is the norm of $\cos(x)$?

Solution. $\langle \cos(x), \cos(x) \rangle = \int_0^{2\pi} \cos(t) \cos(t) \mathrm{d}t = \pi$

Why? There's many ways to evaluate this integral. For instance:

- integration by parts
- using a trig identity
- here's a simple way:
  - $\int_0^{2\pi} \cos^2(t) \mathrm{d}t = \int_0^{2\pi} \sin^2(t) \mathrm{d}t$ ($\cos$ and $\sin$ are just a shift apart)
  - $\cos^2(t) + \sin^2(t) = 1$
  - So: $\int_0^{2\pi} \cos^2(t) \mathrm{d}t = \frac{1}{2} \int_0^{2\pi} 1 \mathrm{d}x = \pi$

Hence, $\cos(x)$ is not normalized. It has norm $\|\cos(x)\| = \sqrt{\pi}$.

Similarly. The same calculation shows that $\cos(kx)$ and $\sin(kx)$ have norm $\sqrt{\pi}$ as well.

Example 183. How do we find, say, b_2?

Solution. Since the functions 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$, ..., the term $b_2\sin(2x)$ is the orthogonal projection of $f(x)$ onto $\sin(2x)$.

In particular, $b_2 = \frac{\langle f(x), \sin(2x)\rangle}{\langle \sin(2x), \sin(2x)\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(2t)\mathrm{d}x$.

In conclusion:

A (nice) $f(x)$ on $[0, 2\pi]$ has the Fourier series

$$f(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

where

$$\begin{aligned} a_k = \frac{\langle f(x), \cos(kx)\rangle}{\langle \cos(kx), \cos(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t, \\ b_k = \frac{\langle f(x), \sin(kx)\rangle}{\langle \sin(kx), \sin(kx)\rangle} &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)\mathrm{d}t, \\ a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t. \end{aligned}$$

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