

## Homework Set 10 (Lecture 34)

### Problem 4

**Example 23.** Find the best approximation of  $f(x) = x^4$  on the interval  $[1, 3]$  using a function of the form  $y = a + bx$ .

**Solution.** First, note that this best approximation is the orthogonal projection of  $x^4$  onto  $\text{span}\{1, x\}$ . However, this orthogonal projection is not simply the projection onto  $1$  plus the projection onto  $x$ . That's because  $1$  and  $x$  are not orthogonal (as functions on  $[1, 3]$ ):

$$\langle 1, x \rangle = \int_1^3 t dt = \left[ \frac{1}{2} t^2 \right]_1^3 = \frac{3^2}{2} - \frac{1}{2} = 4 \neq 0.$$

To find an orthogonal basis for  $\text{span}\{1, x\}$ , following Gram–Schmidt, we compute

$$x - \left( \begin{array}{c} \text{projection of} \\ x \text{ onto } 1 \end{array} \right) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_1^3 t dt}{\int_1^3 1 dt} = x - \frac{4}{2} = x - 2.$$

Hence,  $1, x - 2$  is an orthogonal basis for  $\text{span}\{1, x\}$ .

The desired orthogonal projection of  $x^4$  onto  $\text{span}\{1, x\} = \text{span}\{1, x - 2\}$  therefore is

$$\frac{\langle x^4, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle x^4, x - 2 \rangle}{\langle x - 2, x - 2 \rangle} (x - 2) = \frac{\int_1^3 t^4 dt}{\int_1^3 1 dt} + \frac{\int_1^3 t^4 (t - 2) dt}{\int_1^3 (t - 2)^2 dt} (x - 2)$$

We compute the three new integrals:

$$\begin{aligned} \int_1^3 t^4 dt &= \left[ \frac{1}{5} t^5 \right]_1^3 = \frac{3^5}{5} - \frac{1}{5} = \frac{242}{5} \\ \int_1^3 (t - 2)^2 dt &= \int_1^3 (t^2 - 4t + 4) dt = \left[ \frac{1}{3} t^3 - 2t^2 + 4t \right]_1^3 = 3 - \frac{7}{3} = \frac{2}{3} \\ \int_1^3 t^4 (t - 2) dt &= \int_1^3 (t^5 - 2t^4) dt = \left[ \frac{1}{6} t^6 - \frac{2}{5} t^5 \right]_1^3 = \frac{243}{10} - \left( -\frac{7}{30} \right) = \frac{368}{15} \end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_1^3 t^4 dt}{\int_1^3 1 dt} + \frac{\int_1^3 t^4 (t - 2) dt}{\int_1^3 (t - 2)^2 dt} (x - 2) = \frac{242}{5} + \frac{368}{\frac{2}{3}} (x - 2) = \frac{121}{5} + \frac{184}{5} (x - 2) = \frac{184}{5} x - \frac{247}{5}.$$