Homework Set 10 (Lecture 32)

Problem 1

Example 20. Find the smallest norm solution to $4x_1+3x_2+5x_3=3$.

Solution. If $A=\begin{bmatrix} 4 & 3 & 5 \end{bmatrix}$, then the smallest norm solution is $\boldsymbol{x}=A^+\begin{bmatrix} 3 \end{bmatrix}$.

From earlier computations (see Example 163) we know that $A^+=\frac{1}{4^2+3^2+5^2}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}=\frac{1}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Hence, the smallest norm solution is $\boldsymbol{x}=A^+\begin{bmatrix} 3 \end{bmatrix}=\frac{3}{50}\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Problem 2

Example 21. Determine the best rank 1 approximation of $A=\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution. We first compute the SVD of A:

- First, we need to diagonalize $A^TA=\begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\right)=(2-\lambda)(5-\lambda)-4=\lambda^2-7\lambda+6=(\lambda-1)(\lambda-6)$

  Hence, the eigenvalues of A^TA are $6, 1$.

  - $\lambda=6$: $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \overset{R_2-\frac{1}{2}R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{4}R_1\Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

    Hence, the 6-eigenspace has basis $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

  - $\lambda=1$: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \overset{R_2+2R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

    Hence, the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  Thus $A^TA=PDP^T$ with $D=\begin{bmatrix} 6 & \\ & 1 \end{bmatrix}$ and $P=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA=V\Sigma^2V^T$, we conclude that $V=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\Sigma=\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i=\sigma_i\boldsymbol{u}_i$, we find $\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}=\frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}$.

  For the rank 1 approximation, we only need the first column of U, so we stop here.

Hence, $A=U\Sigma V^T$ with $U=\begin{bmatrix} -5/\sqrt{30} & * & * \\ -2/\sqrt{30} & * & * \\ -1/\sqrt{30} & * & * \end{bmatrix}$, $\Sigma=\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

From the SVD of A, we obtain the best rank 1 approximation by only using the first columns of U and V (and truncating Σ to a 1×1 matrix):

Thus, the best rank 1 approximation of A is $\frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}\begin{bmatrix} \sqrt{6} \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}^T=\sqrt{\frac{6}{30\cdot 5}}\begin{bmatrix} -5 \\ -2 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} 5 & -10 \\ 2 & -4 \\ 1 & -2 \end{bmatrix}$.

Comment. Like for U, we could have omitted the computation of the 1-eigenvector (second column of V).

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