Homework Set 9 (Lecture 31)

Problem 4

Example 17. Determine the pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix}$.

Solution. For such diagonal matrices, we only need to invert the diagonal entries and transpose the dimensions.
$A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/5 \\ 0 & 0 \end{bmatrix}$

Problem 5

Example 18. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$ (without computing the SVD first).

Solution. This matrix clearly has full column rank (because the two columns are not multiples of each other).
Hence, $A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 13 & -6 \\ -6 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 13 & 6 \\ 6 & 13 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ -3 & 2 & 0 \end{bmatrix} = \frac{1}{133} \begin{bmatrix} 8 & 12 & 39 \\ -27 & 26 & 18 \end{bmatrix}$.

Problem 6

Example 19. Determine the pseudoinverse of $A = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$ (by computing the SVD first).

Solution. We first compute the SVD of A:

- First, we need to diagonalize $A^T A = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}$. Let us write $|A|$ for $\det(A)$:

$$\begin{vmatrix} 4-\lambda & -4 & 2 \\ -4 & 4-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = (4-\lambda) \cdot \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - (-4) \cdot \begin{vmatrix} -4 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} -4 & 4-\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (4-\lambda) \cdot (\lambda^2 - 5\lambda) + 4 \cdot (4\lambda) + 2 \cdot (2\lambda) = -\lambda^3 + 9\lambda^2 = \lambda^2(9-\lambda)$$

Hence, the eigenvalues of $A^T A$ are $9, 0, 0$.

  - $\lambda = 9$: $\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \underset{R_3 + \frac{2}{5} R_1 \Rightarrow R_3}{\overset{R_2 - \frac{4}{5} R_1 \Rightarrow R_2}{\rightsquigarrow}} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & -\frac{18}{5} & -\frac{36}{5} \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -5 & -4 & 2 \\ 0 & -\frac{9}{5} & -\frac{18}{5} \\ 0 & 0 & 0 \end{bmatrix}$
$\underset{-\frac{5}{9} R_2 \Rightarrow R_2}{\overset{-\frac{1}{5} R_1 \Rightarrow R_1}{\rightsquigarrow}} \begin{bmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - \frac{4}{5} R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

    Hence, the 9-eigenspace has basis $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

  - $\lambda = 0$: $\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \underset{R_3 - \frac{1}{2} R_1 \Rightarrow R_3}{\overset{R_2 + R_1 \Rightarrow R_2}{\rightsquigarrow}} \begin{bmatrix} 4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4} R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

    Hence, the 0-eigenspace has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

  Thus $A^T A = PDP^T$ with $D = \begin{bmatrix} 9 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{5} \\ -2/3 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 2/\sqrt{5} \end{bmatrix}$.

  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V\Sigma^2 V^T$, we conclude that $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{5} \\ -2/3 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 2/\sqrt{5} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = 1$.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{5} \\ -2/3 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 2/\sqrt{5} \end{bmatrix}$.

Using the SVD of A, we can easily obtain its pseudoinverse:

$$A^+ = V\Sigma^+ U^T = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{5} \\ -2/3 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Comments. This was good practice computing SVDs but we did a lot of work that we could have simplified: Can you see why it was clear that $A^T A$ was going to have 0 as a repeated eigenvalue? Can you see why the last two columns of P are irrelevant in our computation? Can you see how we could have obtained the first column of P without computation? [Also, can you argue geometrically why the pseudoinverse is what it is?]