Homework Set 8 (Lecture 26)

Problem 7

Example 9. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution. We can see right away that $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ is not diagonalizable (because it is a 2×2 Jordan block).

The solution to the differential equation is

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{At} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= e^{5It+Nt} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e^{5It} e^{Nt} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } 5It \text{ and } Nt \text{ commute)} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots \right) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} (1 + Nt) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{(because } N^2 = \mathbf{0}\text{)} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t} & \\ & e^{5t} \end{bmatrix} \begin{bmatrix} 1 - 2t \\ -2 \end{bmatrix} = \begin{bmatrix} (1-2t)e^{5t} \\ -2e^{5t} \end{bmatrix}.
\end{aligned}$$

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