Review. A matrix A has orthonormal columns $\iff$ $A^TA = I$.

Example 73. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(Q)$?

Solution. Recall that, to project onto $\mathrm{col}(A)$, the projection matrix is $P = A(A^TA)^{-1}A^T$.

Since $Q^TQ = I$, to project onto $\mathrm{col}(Q)$, the projection matrix is $P = QQ^T$.

Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q}\cdot\boldsymbol{b})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

Comment. In particular, if Q is not square, then $Q^TQ = I$ but $QQ^T \neq I$. In some sense, QQ^T still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\mathrm{col}(Q)$ it does act like the identity for vectors in $\mathrm{col}(Q)$. (Vectors not in $\mathrm{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\mathrm{col}(Q)$.)

Example 74. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(A)$?

Solution. If A is an invertible $n \times n$ matrix, then $\mathrm{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\mathrm{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Definition 75. An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to be orthonormal, not just orthogonal.]

An $n \times n$ matrix Q is orthogonal $\iff$ $Q^TQ = I$

In other words, $Q^{-1} = Q^T$.

Example 76. What can we say about $\det(Q)$ if Q is orthogonal?

Solution. Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

Review: More on diagonalization

Example 77. (review) If A is a 2×2 matrix with $\det(A) = -8$ and eigenvalue 4. What is the second eigenvalue?

Solution. Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is -2.

$\det(A)$ is the product of the eigenvalues of A.

Why? Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A. We compute the characteristic polynomial $\det(A - \lambda I)$ and determine the λ_i as the roots of that polynomial.

That means that we have the factorization $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_n - \lambda)\cdots(\lambda_n - \lambda)$. Now, set $\lambda = 0$ to conclude that $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Armin Straub
straub@southalabama.edu

Example 78.

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(b) Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so A has eigenvalues $4, -2$.

The 4-eigenspace is $\text{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The -2-eigenspace is $\text{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Important observation. The 4-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the -2-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthogonal!

Review. The product of all eigenvalues $-2 \cdot 4 = -8$ equals the determinant $\det(A) = 1 - 9 = -8$.

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose P so that $P^{-1} = P^T$, which means that P must be **orthogonal** (meaning orthonormal columns). [Choosing such a P is only possible if the eigenspaces of A are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, we then have $A = PDP^T$.

Example 79. (homework) Diagonalize, if possible, the matrices

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We will briefly discuss these next class.