Review. The **projection matrix** for projecting onto $\mathrm{col}(A)$ is $P = A(A^T A)^{-1} A^T$.

Example 58. True or false? If P is the matrix for projecting onto W, then $W = \mathrm{col}(P)$.

Solution. True!

Why? The columns of P are the projections of the standard basis vectors and hence in W. On the other hand, for any vector $\boldsymbol{w}$ in W, we have $P\boldsymbol{w} = \boldsymbol{w}$ so that $\boldsymbol{w}$ is a combination of the columns of P.

[This may take several readings to digest but do read (or ask) until it makes sense!]

In particular. $\mathrm{rank}(P) = \dim W$ (because, for any matrix, $\mathrm{rank}(A) = \dim \mathrm{col}(A)$)

Theorem 59. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

Proof. Suppose that $c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}$. In order to show that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are independent, we need to show that $c_1 = c_2 = \ldots = c_n = 0$.

Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned} 0 &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = c_1\|\boldsymbol{v}_1\|^2 \end{aligned}$$

But $\|\boldsymbol{v}_1\| \neq 0$ and hence $c_1 = 0$. Likewise, we find $c_2 = 0$, …, $c_n = 0$. Hence, the vectors are independent. □

Comment. Note that this result is intuitively obvious: if the vectors were linearly dependent, then one of them could be written as a linear combination of the others. However, all these other vectors (and hence any combination of them) are orthogonal to it.

Orthogonal projections if we have an orthogonal basis

Lemma 60. (orthogonal projection if we have an orthogonal basis)

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthogonal, then the orthogonal projection of $\boldsymbol{w}$ onto $\mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is

$$\hat{\boldsymbol{w}} = \underbrace{\frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_1}} + \ldots + \underbrace{\frac{\boldsymbol{w} \cdot \boldsymbol{v}_n}{\boldsymbol{v}_n \cdot \boldsymbol{v}_n}\boldsymbol{v}_n}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_n}}.$$

Proof. It suffices to show that the error $\boldsymbol{w} - \hat{\boldsymbol{w}}$ is orthogonal to each $\boldsymbol{v}_i$. Indeed:

$$(\boldsymbol{w} - \hat{\boldsymbol{w}}) \cdot \boldsymbol{v}_i = \left(\boldsymbol{w} - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \ldots - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_n}{\boldsymbol{v}_n \cdot \boldsymbol{v}_n}\boldsymbol{v}_n\right) \cdot \boldsymbol{v}_i = \boldsymbol{w} \cdot \boldsymbol{v}_i - \frac{\boldsymbol{w} \cdot \boldsymbol{v}_i}{\boldsymbol{v}_i \cdot \boldsymbol{v}_i}\boldsymbol{v}_i \cdot \boldsymbol{v}_i = 0.$$

□

Important consequence. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

If the $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a basis, but not orthogonal, then we have to solve a system of equations to find the c_i. That is a lot more work than simply computing a few dot products.

Note. In other words, $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector.

Note. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthonormal, then the denominators are all 1.

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Example 61. What is the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$?

Comment. We know how to do this using least squares. (Do it for practice!)
However, realizing that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal makes things easier.
[Actually, here, it is obvious what the projection is going to be if we realized that W is the x-y-plane.]

Solution. (using orthogonality) Because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal, the projection is

$$\underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{projection onto } \boldsymbol{v}_1} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{projection onto } \boldsymbol{v}_2} = \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}.$$

Important note. Note that, at this point, we can easily extend $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to an orthogonal basis of $\mathbb{R}^3$:

That is because the error $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is orthogonal to both of the existing basis vectors.

Therefore $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ is an orthogonal basis of $\mathbb{R}^3$.

This observation underlies the Gram-Schmidt process, which we will discuss next class.

Example 62. Express $\underbrace{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}}_{\boldsymbol{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{v}_3}$.

Solution. Because $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is an orthogonal basis of $\mathbb{R}^3$, we get (much as in the previous example):

$$\begin{aligned} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_1} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_2} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{v}_3} \\ &= \frac{-4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Because we spelled out all the details this looks more involved than it is. We only computed 6 dot products!

Alternative. We could have solved $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ to also find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$.

The numbers are particularly easy here but in general, to find this solution, we have to go through the entire process of Gaussian elimination. On the other hand, if we have an orthogonal basis, the former approach requires less work, because it is just computing a few dot products.

Example 63. Express $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution. This is not an orthogonal basis, so we cannot proceed as in the previous example.

To write $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we need to solve $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

Solving that system (do it!), we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

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