Example 45. A car rental company wants to predict the annual maintenance cost y (in 100USD/year) of a car using the age x (in years) of that car (as an explanatory variable). Based on the observations $(x,\ y) = (2,\ 1),\ (5,\ 2),\ (7,\ 3),\ (8,\ 3)$, predict the cost for a 4.5 year old car (using linear regression).

Solution. Once we compute the regression line $y = a + bx$ (we already did that: $y = \frac{2}{7} + \frac{5}{14}x$), our prediction is $\frac{2}{7} + \frac{5}{14} \cdot 4.5 = \frac{53}{28} \approx 1.89$, that is, 189 USD/year.

Application: multiple linear regression

> *In statistics, **linear regression** is an approach for modeling the relationship between a scalar dependent variable and one or more explanatory variables.*
>
> *The case of one explanatory variable is called simple linear regression.*
>
> *For more than one explanatory variable, the process is called multiple linear regression.*
>
> http://en.wikipedia.org/wiki/Linear_regression

The experimental data might be of the form $(x_i,\ y_i,\ z_i)$, where now the dependent variable z_i depends on two explanatory variables x_i, y_i (instead of just x_i).

Example 46. Set up a linear system to find values for the parameters $a,\ b,\ c$ such that $z = a + bx + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$

Solution. The equations $a + bx_i + cy_i = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\text{design matrix } A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\substack{\text{observation} \\ \text{vector } \boldsymbol{z}}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution by solving $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \boldsymbol{z}$.

Application: Fitting data to other curves

We can also fit the experimental data (x_i, y_i) using other curves.

Example 47. Set up a linear system to find values for the parameters a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits some given points $(x_1, y_1), (x_2, y_2), \ldots$

Solution. $y_i \approx a + bx_i + cx_i^2$ with parameters a, b, c.

The equations $y_i = a + bx_i + cx_i^2$ in matrix form:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\text{design matrix } A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\substack{\text{observation} \\ \text{vector } \boldsymbol{y}}}$$

Again, we determine values for a, b, c by computing a least squares solution to that system.

That is, we need to solve the system $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \boldsymbol{y}$.

Example 48. (extra) Use Sage to find values for a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits the points $(0, 1), (1, 2), (2, 3), (3, -4), (4, -7), (5, -12)$.

Solution. We first input the points:

```
Sage] points = [[0,1],[1,2],[2,3],[3,-4],[4,-7],[5,-12]]
```

We set up the system described in the previous example, then determine a least-squares solution.

```
Sage] X = matrix([[1,0,0],[1,1,1],[1,2,4],[1,3,9],[1,4,16],[1,5,25]])
```

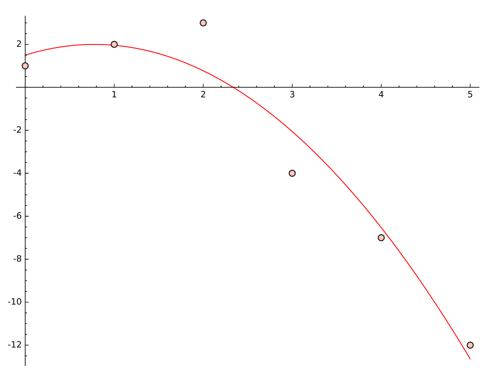
```
Sage] y = vector([1,2,3,-4,-7,-12])
```

```
Sage] (X.transpose()*X).solve_right(X.transpose()*y)
```

$$\left(\frac{3}{2}, \frac{179}{140}, -\frac{23}{28}\right)$$

Hence, the best fitting quadratic curve is $y = \frac{3}{2} + \frac{179}{140} - \frac{23}{28}x^2$. Here's a plot:

```
Sage] scatter_plot(points) + plot(3/2+179/140*x-23/28*x^2,0,5,color='red')
```



Advanced comment. If you are comfortable with Python, you can avoid typing out X and $\boldsymbol{y}$:
[The plot command above now won't work anymore because we are overwriting x with numbers.]

```
Sage] X = matrix([[1,x,x^2] for x,y in points])
```

```
Sage] y = vector([y for x,y in points])
```

More on orthogonality

Projection matrices

The **(orthogonal) projection** $\hat{\boldsymbol{b}}$ of a vector $\boldsymbol{b}$ onto a subspace Y is the vector in Y closest to $\boldsymbol{b}$.

We can compute $\hat{\boldsymbol{b}}$ as follows:

- Write $Y = \operatorname{col}(A)$ for some matrix A.
- Then $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}}$ is a least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$. (i.e. $\hat{\boldsymbol{x}}$ solves $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$)

Why? Why is $A\hat{\boldsymbol{x}}$ the projection of $\boldsymbol{b}$ onto $\operatorname{col}(A)$?

Because, for a least squares solution $\hat{\boldsymbol{x}}$, $A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is as small as possible (and any element in $\operatorname{col}(A)$ is of the form $A\boldsymbol{x}$ for some $\boldsymbol{x}$).

Note. This is a recipe for computing any orthogonal projection! That's because every subspace Y can be written as $\operatorname{col}(A)$ for some choice of the matrix A (take, for instance, A so that its columns are a basis for Y).

Example 49. What is the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$?

Solution. In other words, what is the orthogonal projection of $\boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto $\operatorname{col}(A)$ with $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

In Example 42, we found that the system $A\boldsymbol{x} = \boldsymbol{b}$ has the least squares solution $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ thus is $A\hat{\boldsymbol{x}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$.

Check. The error $\hat{\boldsymbol{b}} - \boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix}$ needs to be orthogonal to $\operatorname{col}(A)$. Indeed: $\begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$.

Example 50. (extra)

(a) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$?

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Solution. (final answer only) The projections are $\left(\frac{11}{6}, \frac{1}{3}, \frac{7}{6}\right)^T$ and $\left(\frac{3}{2}, 0, \frac{3}{2}\right)^T$.