Preparing for the Final

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) The best way to prepare for the final exam is to do the online practice problems for the final.

(b) In addition, review the practice problems for both midterms (for the material up to Midterm #2) as well as the problems below (for the material since Midterm #2).

(c) You can also retake both midterm exams.

Problem 2. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the best rank 1 approximation of A.

(c) Determine the pseudoinverse of A.

(d) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\begin{array}{ccc} 1 & 1 & 0 \end{array}]^T$.)

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\begin{aligned} \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) &= 0 - 1 \cdot \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix}\right) + (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) \\ &= -(2-\lambda) + (2-\lambda)\underbrace{((2-\lambda)(1-\lambda)-1)}_{=\lambda^2-3\lambda+1} \\ &= (2-\lambda)(\lambda^2-3\lambda) = (2-\lambda)\lambda(\lambda-3). \end{aligned}$$

Hence, the eigenvalues are $0, 2, 3$.

- The 0-eigenspace null$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

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- The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Next, $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only) is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The pseudoinverse of A is

$$A^+ = V\Sigma^+ U^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}.$$

(d) The smallest solution to $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{x} = A^+\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}.$$

(For comparison, $\|\boldsymbol{x}\| = \sqrt{11/6} \approx 1.354$ is indeed less than $\|\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T\| = \sqrt{2} \approx 1.414$.)

Problem 3.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Determine the best rank 1 approximation of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(c) Determine the SVD of $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution.

(a) $A^TA = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ has 6-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 4-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{u}_3$ needs to be chosen so that the matrix U is orthogonal. To find such a vector, we can start with a random vector like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and then apply a step of Gram–Schmidt to produce a vector that is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$:

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. We normalize this to $\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only) is

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} \sqrt{6} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(c) $A^TA = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(Hint: We can immediately read off the 0-eigenvector (make sure that's obvious!). It then follows from the spectral theorem that the vector orthogonal to it must be another eigenvector.)

Normalizing, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2\boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

Let's choose $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the only other choice is $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$). Then, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Problem 4. Find the best approximation of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

Solution. The best approximation we are looking for is the orthogonal projection of $f(x)$ onto span$\{1, \sqrt{x}\}$, where the dot product of functions is

$$\langle f, g \rangle = \int_0^4 f(t)g(t)\mathrm{d}t.$$

To find an orthogonal basis for $\operatorname{span}\{1,\sqrt{x}\}$, following Gram–Schmidt, we compute

$$\sqrt{x}-\begin{pmatrix}\text{projection of}\\ \sqrt{x}\text{ onto }1\end{pmatrix}=\sqrt{x}-\frac{\langle\sqrt{x},1\rangle}{\langle 1,1\rangle}1=\sqrt{x}-\frac{4}{3}.$$

In the last step, we used that

$$\langle 1,1\rangle=\int_0^4 1\mathrm{d}t=4,\quad \langle\sqrt{x},1\rangle=\int_0^4\sqrt{t}\mathrm{d}t=\left[\frac{1}{3/2}t^{3/2}\right]_0^4=\frac{16}{3}.$$

Hence, $1,\sqrt{x}-\frac{4}{3}$ is an orthogonal basis for $\operatorname{span}\{1,\sqrt{x}\}$.

The orthogonal projection of $f\colon[0,4]\to\mathbb{R}$ onto $\operatorname{span}\{1,\sqrt{x}\}=\operatorname{span}\left\{1,\sqrt{x}-\frac{4}{3}\right\}$ therefore is

$$\frac{\langle f,1\rangle}{\langle 1,1\rangle}1+\frac{\left\langle f,\sqrt{x}-\frac{4}{3}\right\rangle}{\left\langle\sqrt{x}-\frac{4}{3},\sqrt{x}-\frac{4}{3}\right\rangle}\left(\sqrt{x}-\frac{4}{3}\right)\ =\ \frac{1}{4}\int_0^4 f(t)\mathrm{d}t+\frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\int_0^4 f(t)\left(\sqrt{t}-\frac{4}{3}\right)\mathrm{d}t.$$

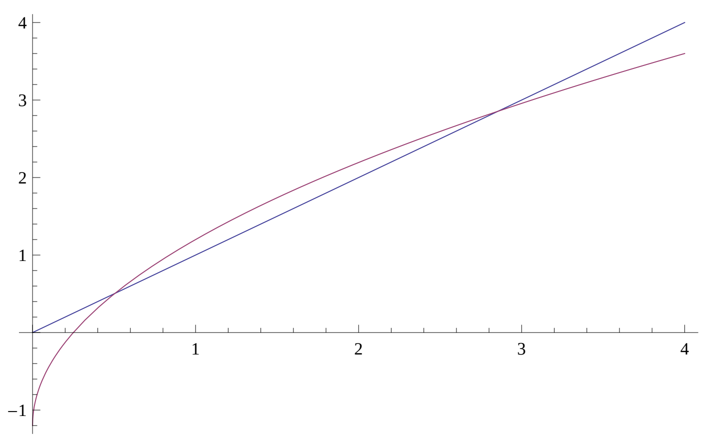
Here, we used that

$$\left\langle\sqrt{x}-\frac{4}{3},\sqrt{x}-\frac{4}{3}\right\rangle=\int_0^4\left(\sqrt{t}-\frac{4}{3}\right)^2\mathrm{d}t=\int_0^4\left(t-\frac{8}{3}\sqrt{t}+\frac{16}{9}\right)\mathrm{d}t=\left[\frac{t^2}{2}-\frac{16}{9}t^{3/2}+\frac{16}{9}t\right]_0^4=8-\frac{128}{9}+\frac{64}{9}=\frac{8}{9}.$$

In our case, this best approximation is

$$\begin{aligned}&\frac{1}{4}\int_0^4 t\mathrm{d}t+\frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\int_0^4 t\left(\sqrt{t}-\frac{4}{3}\right)\mathrm{d}t\\ =\ &\frac{1}{4}\left[\frac{t^2}{2}\right]_0^4+\frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\left[\frac{2}{5}t^{5/2}-\frac{2}{3}t^2\right]_0^4=2+\frac{12}{5}\left(\sqrt{x}-\frac{4}{3}\right)=\frac{12}{5}\sqrt{x}-\frac{6}{5}.\end{aligned}$$

The plot below confirms the quality of this approximation:



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Problem 5.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has [ ].

(b) A is singular if and only if [$\dim \operatorname{null}(A)$].

(c) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* =$ [ ].

(d) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\boldsymbol{v}\| =$ [ ].

(e) By Euler's identity, $e^{ix} =$ [ ].

(f) What exactly does it mean for a matrix A to have full column rank?

(g) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ =$ [ ].

(h) If A is invertible then its pseudoinverse is $A^+ =$ [ ].

(i) If A has full column rank then its pseudoinverse is $A^+ =$ [ ].

(j) Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? [ ]

(k) Write down the 2×2 rotation matrix by angle θ.

Solution.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has λ^3-eigenvalue $\boldsymbol{v}$.

(b) A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A) > 0$.

(c) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* = \begin{bmatrix} -i & 3 & -5i \\ 1-2i & 4 & 6+i \end{bmatrix}$.

(d) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\boldsymbol{v}\| = \sqrt{|1-i|^2 + 2^2} = \sqrt{6}$.

(e) By Euler's identity, $e^{ix} = \cos(x) + i\sin(x)$.

(f) A matrix A has full column rank if its rank equals the number of columns.

(g) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/7 \\ 0 & 0 \end{bmatrix}$.

(h) If A is invertible, then $A^+ = A^{-1}$.

(i) If A has full column rank then its pseudoinverse is $A^+ = (A^T A)^{-1} A^T$.

(j) The one of smallest norm.

(k) The 2×2 rotation matrix by angle θ is $\left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$.

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