Assessment Quiz #2

MATH 316 — Linear Algebra II
Friday, March 15

Please print your name:

Problem 1. (4 points) Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ on the extra sheet.

The solution is $\boldsymbol{y}(t) = \boxed{}$.

Make sure to check your answer by plugging into the differential equation! There will be no partial credit and you have plenty of time.

Solution.

- $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(1-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

  Hence, the eigenvalues of A are $-1, 2$.

  The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{bmatrix}} \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\frac{1}{3}\begin{bmatrix} -1 \\ 4 \end{bmatrix}} = \frac{1}{3}\begin{bmatrix} -e^{-t} + 4e^{2t} \\ e^{-t} + 8e^{2t} \end{bmatrix} \end{aligned}$$

□

Problem 2. (1+3+1 points) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = 1$, $a_1 = 7$.

(a) The next two terms are $a_2 = \boxed{}$ and $a_3 = \boxed{}$.

(b) A Binet-like formula for a_n is $a_n = \boxed{}$.

(c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \boxed{}$.

Again, work on the extra sheet and be sure to check your answer to (b) by comparing with the values in (a).

Solution.

(a) $a_2 = 17$, $a_3 = 2 \cdot 17 + 3 \cdot 7 = 55$

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(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\left[\begin{smallmatrix} 2 & 3 \\ 1 & 0 \end{smallmatrix}\right]$ are $3, -1$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = 3\alpha_1 - \alpha_2 = 7$.

Solving, we find $\alpha_1 = 2$ and $\alpha_2 = -1$ so that, in conclusion, $a_n = 2 \cdot 3^n - (-1)^n$.

Comment. Alternatively, we could have proceeded as we did in the case of the Fibonacci numbers: starting with the recursion matrix T, we compute its diagonalization $T = PDP^{-1}$. Multiplying out $PD^nP^{-1}\left[\begin{smallmatrix} a_1 \\ a_0 \end{smallmatrix}\right]$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$. □

Problem 3. (1+1+2+2 points) Fill in the blanks.

(a) If $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, then $e^{At} =$ [ ].

(b) An example of a 2×2 matrix that is not diagonalizable is [ ].

(c) If A has eigenvalue 3, then A^2 has eigenvalue [ ], $4A$ eigenvalue [ ], and A^T eigenvalue [ ].

(d) How many different Jordan normal forms are there in the following cases?

- A 5×5 matrix with eigenvalues $1, 1, 2, 2, 2$? [ ]
- A 9×9 matrix with eigenvalues $1, 1, 2, 2, 2, 4, 4, 4, 4$? [ ]

Solution.

(a) If $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$, then $e^{At} = \begin{bmatrix} e^{3t} & \\ & e^{-t} \end{bmatrix}$.

(b) An example of a 2×2 matrix that is not diagonalizable is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (This is a Jordan block!)

(c) If A has eigenvalue 3, then A^2 has eigenvalue $3^2 = 9$, $4A$ eigenvalue $4 \cdot 3 = 12$, and A^T eigenvalue 3.

(d) $2 \cdot 3 = 6$ and $2 \cdot 3 \cdot 5 = 30$ different Jordan normal forms. □