Preparing for Midterm #2

MATH 316 — Linear Algebra II
Wednesday, Mar 27

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix}$ has characteristic polynomial $(4-\lambda)(6-\lambda) - 8 = \lambda^2 - 10\lambda + 16 = (\lambda - 2)(\lambda - 8)$.

  Hence, the eigenvalues of A are $2, 8$.

  The 8-eigenspace $\operatorname{null}\left(\begin{bmatrix} -4 & -8 \\ -1 & -2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{8t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} -2e^{8t} & 4e^{2t} \\ e^{8t} & e^{2t} \end{bmatrix}} \underbrace{\left(-\frac{1}{6}\right)\begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\frac{1}{6}\begin{bmatrix} 1 \\ 5 \end{bmatrix}} = \frac{1}{6}\begin{bmatrix} 20e^{2t} - 2e^{8t} \\ 5e^{2t} + e^{8t} \end{bmatrix} \end{aligned}$$

□

Problem 2.

(a) Convert the third-order differential equation

$$y''' = 6y'' - 3y' - 10y, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

to a system of first-order differential equations.

(b) Solve the original differential equation by solving the system.

Solution.

(a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 6y'' - 3y' - 10y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -10y_1 - 3y_2 + 6y_3 \end{cases}$.

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In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, to compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}$, we need to diagonalize A.

  - The eigenvalues of A are $\lambda = 5, 2, -1$.
  - The 5-eigenspace $\operatorname{null}\left(\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & -3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$.
  - The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -10 & -3 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -3 & 7 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & -1 \end{bmatrix}$.

- Then, we compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned}
\boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix} \\
&= \frac{1}{18} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} -e^{5t} \\ 20e^{2t} \\ -e^{-t} \end{bmatrix} \\
&= \frac{1}{18} \begin{bmatrix} -e^{5t} + 20e^{2t} - e^{-t} \\ -5e^{5t} + 40e^{2t} + e^{-t} \\ -25e^{5t} + 80e^{2t} - e^{-t} \end{bmatrix}
\end{aligned}$$

  In particular, the original differential equation is solved by $y(t) = \frac{1}{18}(-e^{5t} + 20e^{2t} - e^{-t})$.

  Comment. To compute $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$, we solve $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to find $\boldsymbol{x} = \frac{1}{18}\begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$.

  Next comment. Obviously, computations will be more pleasant on the exam.

□

Problem 3.

(a) What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $7, 7, 3, 3, 3, 3$?

(b) How many different Jordan normal forms are there for a 10×10 matrix with eigenvalues $8, 6, 6, 2, 2, 2, 1, 1, 1, 1$?

Solution.

(a) There are $2 \cdot 5 = 10$ possibilities:

$$\begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 &&& \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix},$$

$$\begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 &&& \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}$$

(b) There are $1 \cdot 2 \cdot 3 \cdot 5 = 30$ possible different Jordan normal forms. ☐

Problem 4. Consider the sequence a_n defined by $a_{n+2} = 4a_{n+1} - a_n$ and $a_0 = 1$, $a_1 = 0$.

(a) Determine the next three terms.

(b) A Binet-like formula for a_n is $a_n =$ ☐.

(c) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ☐.

Again, work on the extra sheet and be sure to check your answer to (b) by comparing with the values in (a).

Solution.

(a) $a_2 = -1$, $a_3 = -4$, $a_4 = -15$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}$ are $2 + \sqrt{3} \approx 3.732$ and $2 - \sqrt{3} \approx 0.268$.

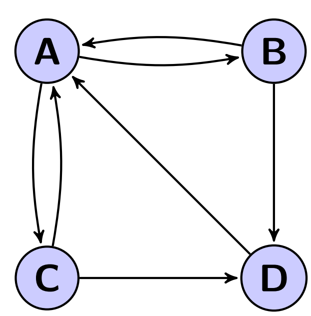
Hence, $a_n = \alpha_1 (2 + \sqrt{3})^n + \alpha_2 (2 - \sqrt{3})^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = (2 + \sqrt{3})\alpha_1 + (2 - \sqrt{3})\alpha_2 = 0$.

Solving, we find $\alpha_1 = \frac{\sqrt{3} - 2}{2\sqrt{3}}$ and $\alpha_2 = \frac{\sqrt{3} + 2}{2\sqrt{3}}$ so that, in conclusion, $a_n = \frac{\sqrt{3} - 2}{2\sqrt{3}}(2 + \sqrt{3})^n + \frac{\sqrt{3} + 2}{2\sqrt{3}}(2 - \sqrt{3})^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2 + \sqrt{3}$. ☐

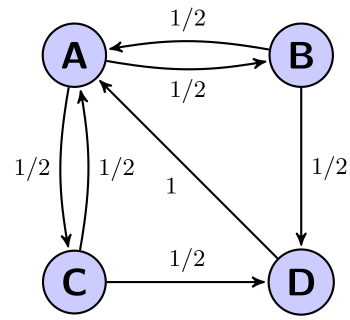
Problem 5. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Solution. Recall that we model a random surfer, who randomly clicks on links. Let a_t be the probability that such a surfer will be on page A at time t. Likewise, b_t, c_t, d_t are the probabilities that the surfer will be on page B, C or D.

The transition probabilities are indicated in the diagram to the right.



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 1 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 0 \cdot d_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{=T} \begin{bmatrix} a_t \\ b_t \\ c_t \\ d_t \end{bmatrix}$$

To find the equilibrium state, we determine an appropriate 1-eigenvector of the transition matrix T.

The 1-eigenspace is $\operatorname{null}(T - 1 \cdot I) = \operatorname{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}\right)$

To compute a basis, we perform Gaussian elimination (details below): $\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. (Note that its entries add up to $2+1+1+1=5$.)

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$. This is the PageRank vector.

Correspondingly, we rank A the highest, followed by B, C, D which we rank equally.

[In hindsight, can you (at least sort of) see, directly from the diagram, why the PageRank is what it is?]

The full steps of the Gaussian elimination are:

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_2 + \frac{1}{2}R_1 \Rightarrow R_2 \\ R_3 + \frac{1}{2}R_1 \Rightarrow R_3}} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_3 + \frac{1}{3}R_2 \Rightarrow R_3 \\ R_4 + \frac{2}{3}R_2 \Rightarrow R_4}} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \underset{\rightsquigarrow}{R_4 + R_3 \Rightarrow R_4} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\rightsquigarrow}{\substack{-1 R_1 \Rightarrow R_1 \\ -\frac{4}{3}R_2 \Rightarrow R_2 \\ -\frac{3}{2}R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{\substack{R_1 + \frac{1}{2}R_3 \Rightarrow R_1 \\ R_2 + \frac{1}{3}R_3 \Rightarrow R_2}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\rightsquigarrow}{R_1 + \frac{1}{2}R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This was good practice of elimination! However, notice that we can actually find an eigenvector $\boldsymbol{x}$ with less effort by spelling out the equations: for instance, the second one is just $\frac{1}{2}x_1 - x_2 = 0$. Do that! $\square$

Problem 6. Determine an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

(a) If A is the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) If A is the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

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Solution. In each case, let W be the plane spanned by the given two vectors. Then A has 1-eigenspace W and -1-eigenspace $W^\perp$. We need orthonormal bases for W and $W^\perp$ in order to write down the diagonalization $A = PDP^T$.

Important comment. Note that, if we just use any bases for W and $W^\perp$, then we would only get a diagonalization of the type $A = PDP^{-1}$.

(a) Here, $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

- (basis for W) Since the vectors are already orthogonal, we normalize to find that $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for W.

- (basis for $W^\perp$) We can read off that $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is an orthonormal basis for $W^\perp$.

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & 0 & -1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) Here, $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

- (basis for W) We apply Gram–Schmidt:

  $$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

  Normalizing, we find that $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ is an orthonormal basis for W.

- (basis for $W^\perp$) Since $W = \operatorname{col}\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)$, we have $W^\perp = \operatorname{null}\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right)$.

  Solving the system, $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$, we find that $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ is a basis for $W^\perp$. Normalized: $\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & -1/\sqrt{6} \\ 0 & 5/\sqrt{30} & -1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. □

Problem 7.

(a) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are $\boxed{\quad}$.

What are the eigenspaces of that matrix?

(b) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) = \boxed{\quad}$, and the eigenvalues of A are $\boxed{\quad}$. What are the eigenspaces of A?

(c) Precisely state the spectral theorem.

(d) If A is a reflection matrix, then $A^{-1} = \boxed{\quad}$.

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(e) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $A^n =$ [ ] and $e^{At} =$ [ ].

(f) If $N^4 = \mathbf{0}$, then $e^{Nt} =$ [ ].

Solution.

(a) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are $1, 1, 1, 0, 0$. The 1-eigenspace is the 3-dimensional subspace that we're projecting on, and the 0-eigenspace is the (2-dimensional) orthogonal complement of that subspace.

(b) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) = -1$, and the eigenvalues of A are 1, 1, -1. The 1-eigenspace is the plane that A is reflecting through, and the -1-eigenspace is the normal direction of the plane (the orthogonal complement).

(c) A symmetric (real) matrix A can always be diagonalized. Moreover, all eigenvalues are real and the eigenspaces are orthogonal.

Alternatively: Every symmetric $n \times n$ matrix A can be decomposed as $A = PDP^T$, where D is a diagonal matrix and P is orthogonal.

(d) If A is a reflection matrix, then $A^{-1} = A$ (because $A^2 = I$).

(e) $A^n = \begin{bmatrix} 2^n & & \\ & 3^n & \\ & & 4^n \end{bmatrix}$, $e^{At} = \begin{bmatrix} e^{2t} & & \\ & e^{3t} & \\ & & e^{4t} \end{bmatrix}$

(f) If $N^4 = \mathbf{0}$, then $e^{Nt} = I + Nt + \frac{1}{2}N^2t^2 + \frac{1}{6}N^3t^3$. □