Preparing for Midterm #1

MATH 316 — Linear Algebra II
Friday, Feb 15

Please print your name:

Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W.

(c) Determine the QR decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(d) Determine a basis for the orthogonal complement $W^{\perp}$.

Problem 2.

(a) Find the least squares solution to the system $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto the space $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(-2, 1), (-1, 0), (0, 3), (2, 1)$.

(d) Determine the projection matrix P for orthogonally projecting onto W.

Problem 3. A scientist tries to find the relation between the mysterious quantities x and y.

She measures the following values:

x	1	2	3	4
y	2	5	9	17

(a) Our scientist has reason to expect that y is a linear function of the form $a + bx$. Find the best estimate for the coefficients. ["best" in the least squares sense]

(b) What changes if we suppose that y is a quadratic function of the form $a + bx + cx^2$? Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Problem 4.

(a) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ as $A = PDP^T$. (That is, find the matrices P and D.)

(b) Let A be a symmetric 2×2 matrix with 2-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\det(A) = -6$. Diagonalize A.

Armin Straub
straub@southalabama.edu

Problem 5.

(a) Is it true that $A^T A$ is always symmetric?

(b) When is $A^T A$ a diagonal matrix?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices A is it true that $A^{-1} = A^T$?

Problem 6.

(a) We want to find values for the parameters a, b, c such that $y = a + bx + \frac{c}{x}$ best fits some given points (x_1, y_1), (x_2, y_2), ... Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

(b) We want to find values for the parameters a, b such that $y = (a + bx)e^x$ best fits some given points (x_1, y_1), (x_2, y_2), ... Set up a linear system such that $[a, b]^T$ is a least squares solution.

(c) We want to find values for the parameters a, b, c such that $z = a + bx - c\sqrt{y}$ best fits some given points (x_1, y_1, z_1), (x_2, y_2, z_2), ... Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Problem 7. Let W be the subspace of $\mathbb{R}^4$ of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for W.

(b) Find a basis for the orthogonal complement $W^\perp$.

(c) Compute the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto $W^\perp$.

(d) Find $\boldsymbol{b}_1$ in W and $\boldsymbol{b}_2$ in $W^\perp$ such that $\boldsymbol{b}_1 + \boldsymbol{b}_2 = (1, 1, 1, 1)^T$.

Problem 8. Suppose that A is a 3×5 matrix of rank 3.

(a) For each of the four fundamental subspaces of A, state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\operatorname{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.

Armin Straub
straub@southalabama.edu