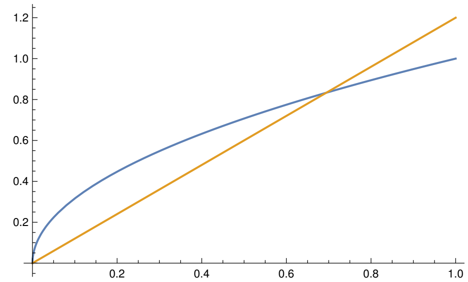
Example 164. Find the best approximation of $f(x)=\sqrt{x}$ on the interval $[0,1]$ using a function of the form $y=ax$.

Solution. The orthogonal projection of $f\colon [0,1]\to\mathbb{R}$ onto $\operatorname{span}\{x\}$ is

$$\frac{\langle f,x\rangle}{\langle x,x\rangle}x=\frac{\int_0^1 f(t)t\mathrm{d}t}{\int_0^1 t^2\mathrm{d}t}x=3x\int_0^1 t f(t)\mathrm{d}t.$$

In our case, the best approximation is

$$3x\int_0^1 t\sqrt{t}\mathrm{d}t=3x\int_0^1 t^{3/2}\mathrm{d}t=3x\left[\frac{1}{5/2}t^{5/2}\right]_0^1=\frac{6}{5}x.$$



Example 165. Find the best approximation of $f(x)=\sqrt{x}$ on the interval $[0,1]$ using a function of the form $y=a+bx$.

Important observation. The orthogonal projection of $f\colon [0,1]\to\mathbb{R}$ onto $\operatorname{span}\{1,x\}$ is not simply the projection onto 1 plus the projection onto x. That's because 1 and x are not orthogonal:

$$\langle 1,x\rangle=\int_0^1 t\mathrm{d}t=\frac{1}{2}\neq 0.$$

Solution. To find an orthogonal basis for $\operatorname{span}\{1,x\}$, following Gram–Schmidt, we compute

$$x-\begin{pmatrix}\text{projection of}\\ x \text{ onto } 1\end{pmatrix}=x-\frac{\langle x,1\rangle}{\langle 1,1\rangle}1=x-\frac{1}{2}.$$

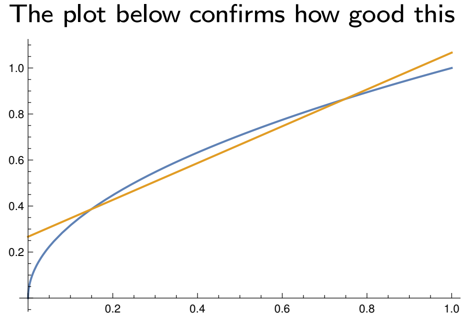
Hence, $1, x-\frac{1}{2}$ is an orthogonal basis for $\operatorname{span}\{1,x\}$.
The orthogonal projection of $f\colon[0,1]\to\mathbb{R}$ onto $\operatorname{span}\{1,x\}=\operatorname{span}\left\{1,x-\frac{1}{2}\right\}$ therefore is

$$\frac{\langle f,1\rangle}{\langle 1,1\rangle}1+\frac{\left\langle f,x-\frac{1}{2}\right\rangle}{\left\langle x-\frac{1}{2},x-\frac{1}{2}\right\rangle}\left(x-\frac{1}{2}\right) = \int_0^1 f(t)\mathrm{d}t+\frac{\int_0^1 f(t)\left(t-\frac{1}{2}\right)\mathrm{d}t}{\int_0^1\left(t-\frac{1}{2}\right)^2\mathrm{d}t}\left(x-\frac{1}{2}\right)$$
$$= \int_0^1 f(t)\mathrm{d}t+(12x-6)\int_0^1 f(t)\left(t-\frac{1}{2}\right)\mathrm{d}t.$$

In our case, this best approximation is

$$\int_0^1\sqrt{t}\mathrm{d}t+(12x-6)\int_0^1\sqrt{t}\left(t-\frac{1}{2}\right)\mathrm{d}t = \left[\frac{1}{3/2}t^{3/2}\right]_0^1+(12x-6)\left[\frac{1}{5/2}t^{5/2}-\frac{1}{2}\frac{1}{3/2}t^{3/2}\right]_0^1$$
$$= \frac{2}{3}+(12x-6)\left(\frac{2}{5}-\frac{1}{3}\right)=\frac{4}{5}\left(x+\frac{1}{3}\right)$$

The plot below confirms how good this linear approximation is (compare with the previous example):



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