Review. complex numbers, fundamental theorem of algebra

Example 141. We can identify complex numbers $x+iy$ with vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$. Then, what is the geometric effect of multiplying with i?

Solution. Algebraically, the effect of multiplying $x+iy$ with i obviously is $i(x+iy)=-y+ix$.
Since multiplication with i is obviously linear, we can represent it using a 2×2 matrix J acting on vectors $\begin{bmatrix} x \\ y \end{bmatrix}$.
$J\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (this is the same as saying $i\cdot 1=i$) and $J\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (this is the same as saying $i\cdot i=-1$).
Hence, $J=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This is precisely the rotation matrix for a rotation by $90°$.
In other words, multiplication with i has the geometric effect of rotating complex numbers by $90°$.
Comment. The relation $i^2=-1$ translates to $J^2=-I$.
Complex numbers as 2×2 matrices. In light of the above, we can express complex numbers $x+iy$ as the 2×2 matrix $xI+yJ=\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. Adding and multiplying these matrices behaves exactly the same way as adding or multiplying the complex numbers directly.
For instance, $(2+3i)(4-i)=8+10i-3i^2=11+10i$ versus $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}=\begin{bmatrix} 11 & -10 \\ 10 & 11 \end{bmatrix}$.
Likewise for inverses: $\frac{1}{2+3i}=\frac{2-3i}{(2+3i)(2-3i)}=\frac{2-3i}{13}$ versus $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}^{-1}=\frac{1}{13}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

Example 142. (extra) Find a unitary matrix Q whose first column is a multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Solution. We need to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $\begin{bmatrix} 1 \\ i \end{bmatrix}^*\begin{bmatrix} a \\ b \end{bmatrix}=a-ib=0$. Choose, say, $a=i$, $b=1$.
This leads to the unitary matrix $Q=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$. Indeed, $Q^*Q=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

More details on the spectral theorem

Let us add $\langle \boldsymbol{v},\boldsymbol{w}\rangle$ to our notations for the dot product: $\langle \boldsymbol{v},\boldsymbol{w}\rangle=\boldsymbol{v}^T\boldsymbol{w}=\boldsymbol{v}\cdot\boldsymbol{w}$.

- In our story of orthogonality, the important player has been the dot product. However, one could argue that the fundamental quantity is actually the norm:
  $\langle \boldsymbol{v},\boldsymbol{w}\rangle=\frac{1}{4}(\|\boldsymbol{v}+\boldsymbol{w}\|^2-\|\boldsymbol{v}-\boldsymbol{w}\|^2)$. See Example 18.

- Accepting the dot product as immensely important, we see that symmetric matrices (i.e. matrices A such that $A=A^T$) are of interest.
  For any matrix A, $\langle A\boldsymbol{v},\boldsymbol{w}\rangle=\langle \boldsymbol{v},A^T\boldsymbol{w}\rangle$.
  It follows that, a matrix A is symmetric if and only if $\langle A\boldsymbol{v},\boldsymbol{w}\rangle=\langle \boldsymbol{v},A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v},\boldsymbol{w}$.

- Similarly, let Q be an orthogonal matrix (i.e. Q is a square matrix with $Q^TQ=I$).
  Then, $\langle Q\boldsymbol{v},Q\boldsymbol{w}\rangle=\langle \boldsymbol{v},\boldsymbol{w}\rangle$.
  In fact, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v},A\boldsymbol{w}\rangle=\langle \boldsymbol{v},\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v},\boldsymbol{w}$.
  Comment. We observed in Example 134 that orthogonal matrices Q correspond to rotations ($\det Q=1$) or reflections ($\det Q=-1$) [or products thereof]. The equality $\langle Q\boldsymbol{v},Q\boldsymbol{w}\rangle=\langle \boldsymbol{v},\boldsymbol{w}\rangle$ encodes the fact that these types (and only these!) of geometric transformations preserve angles and lengths.

Armin Straub
straub@southalabama.edu

(Spectral theorem)

A $n\times n$ matrix A is symmetric if and only if it can be decomposed as $A=PDP^T$, where

- D is a diagonal matrix, $(n\times n)$

The diagonal entries λ_i are the **eigenvalues** of A.

- P is orthogonal. $(n\times n)$

The columns of P are **eigenvectors** of A.

Note that, in particular, A is always diagonalizable, the eigenvalues (and hence, the eigenvectors) are all real, and, most importantly, the eigenspaces of A are orthogonal.

The "only if" part says that, if A is symmetric, then we get a diagonalization $A=PDP^T$. The "if" part says that, if $A=PDP^T$, then A is symmetric (which follows from $A^T=(PDP^T)^T=(P^T)^TD^TP^T=PDP^T=A$).

Let us prove the following important parts of the spectral theorem.

Theorem 143.

(a) If A is symmetric, then the eigenspaces of A are orthogonal.

(b) If A is real and symmetric, then the eigenvalues of A are real.

Proof.

(a) We need to show that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle\boldsymbol{v},\boldsymbol{w}\rangle=\boldsymbol{0}$.
Suppose that $A\boldsymbol{v}=\lambda\boldsymbol{v}$ and $A\boldsymbol{w}=\mu\boldsymbol{w}$ with $\lambda\neq\mu$.
Then, $\lambda\langle\boldsymbol{v},\boldsymbol{w}\rangle=\langle\lambda\boldsymbol{v},\boldsymbol{w}\rangle=\langle A\boldsymbol{v},\boldsymbol{w}\rangle=\langle\boldsymbol{v},A^T\boldsymbol{w}\rangle=\langle\boldsymbol{v},A\boldsymbol{w}\rangle=\langle\boldsymbol{v},\mu\boldsymbol{w}\rangle=\mu\langle\boldsymbol{v},\boldsymbol{w}\rangle$.
However, since $\lambda\neq\mu$, $\lambda\langle\boldsymbol{v},\boldsymbol{w}\rangle=\mu\langle\boldsymbol{v},\boldsymbol{w}\rangle$ is only possible if $\langle\boldsymbol{v},\boldsymbol{w}\rangle=0$.

(b) Suppose λ is a nonreal eigenvalue with nonzero eigenvector $\boldsymbol{v}$. Then, $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector and, since $\lambda\neq\bar{\lambda}$, we have two eigenvectors with different eigenvalues. By the first part, these two eigenvectors must be orthogonal in the sense that $\bar{\boldsymbol{v}}^T\boldsymbol{v}=0$. But $\bar{\boldsymbol{v}}^T\boldsymbol{v}=\boldsymbol{v}^*\boldsymbol{v}=\|\boldsymbol{v}\|^2\neq0$. This shows that it is impossible to have a nonzero eigenvector for a nonreal eigenvalue. □

Let us highlight the following point we used in our proof:

Let A be a real matrix. If $\boldsymbol{v}$ is a λ-eigenvector, then $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector.

See, for instance, Example 78. This is just a consequence of the basic fact that we cannot algebraically distinguish between $+i$ and $-i$.