Example 135. As in the previous example, let Q_θ be the 2×2 matrix for rotation by angle θ in the plane. What is $Q_\alpha Q_\beta$?

Solution. Note that $Q_\alpha Q_\beta \boldsymbol{x}$ first rotates $\boldsymbol{x}$ by angle β and then by angle α. For geometric reasons, it is obvious that this is the same as if we rotated $\boldsymbol{x}$ by $\alpha + \beta$. It follows that $Q_\alpha Q_\beta = Q_{\alpha+\beta}$.

Comment. This allows us to derive interesting trig identities:

$$\begin{aligned} Q_\alpha Q_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \ldots \\ \ldots & \ldots \end{bmatrix} \\ Q_{\alpha+\beta} &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \end{aligned}$$

It follows that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Comment. If we set $\beta = \alpha$, this simplifies to $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$, the double angle formula that you have probably used countless times in Calculus.

Comment. Similarly, we find an identity for $\sin(\alpha+\beta)$. Spell it out!

More on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at Example 78.
- The **absolute value** of the complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.

  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 136.

- For any matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.)

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Example 137. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1+i & 2-3i \end{bmatrix} \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} = |1-i|^2 + |2+3i|^2 = 2+13$. Hence, $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 138. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

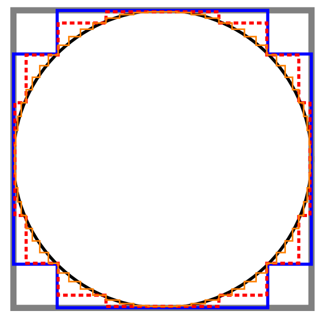
Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Example 139. What is $\frac{1}{2+3i}$?

Solution. $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$.

In general. $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

Remark 140. (April Fools' Day!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

Comment. We'll actually come back to this. It's related to linear algebra in infinite dimensions.