Example 131. Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- We first diagonalize $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
  - $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so the eigenvalues are ± 1.
  - The 1-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}}_{=\begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}}\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix} \end{aligned}$$

Check. Indeed, $y_1 = \frac{1}{2}(e^t + e^{-t})$ and $y_2 = \frac{1}{2}(e^t - e^{-t})$ satisfy the system of differential equations $y_1' = y_2$ and $y_2' = y_1$ as well as the initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

Comment. You have actually met these functions in Calculus! $y_1 = \cosh(t)$ and $y_2 = \sinh(t)$. Check out the next example for the connection to $\cos(t)$ and $\sin(t)$.

Example 132.

(a) Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Show that $\boldsymbol{y} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ solves the same IVP. What do you conclude?

Solution.

(a) From Example 137, we know that $A = PDP^{-1}$ with $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.
The system is therefore solved by:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} \\ -e^{-it} \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} ie^{it} + ie^{-it} \\ e^{it} - e^{-it} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix} \end{aligned}$$

(b) Clearly, $\boldsymbol{y}(0) = \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. On the other hand, $y_1' = -\sin(t) = -y_2$ and $y_2' = \cos(t) = y_1$, so that $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$. Since the solution to the IVP is unique, it follows that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix}$.
We have just discovered **Euler's identity**!

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Theorem 133. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Another short proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Rotation matrices

Example 134. Write down a 2×2 matrix Q for rotation by angle θ in the plane.

Comment. Why should we even be able to represent something like rotation by a matrix? Meaning that $Q\boldsymbol{x}$ should be the vector $\boldsymbol{x}$ rotated by θ. Recall from Linear Algebra I that every **linear map** can be represented by a matrix. Then think about why rotation is a linear map.

Solution. We can determine Q by figuring out $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the first column of Q) and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the second column of Q).

Since $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, we conclude that $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Comment. Note that we don't need previous knowledge of $\cos$ and $\sin$. We could have introduced these trig functions on the spot.

Comment. Note that it is geometrically obvious that Q is orthogonal. (Why?)

It is clear that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = 1$. Noting that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = \cos^2\theta + \sin^2\theta$, we have rediscovered Pythagoras.

Advanced comment. Actually, every orthogonal 2×2 matrix Q with $\det(Q) = 1$ is a rotation by some angle θ. Orthogonal matrices with $\det(Q) = -1$ are reflections.